MATHCOUNTS®

2018 State Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2018 MATHCOUNTS® State Competition. These solutions provide creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions
with the MATHCOUNTS community.

2018 State Competition Sprint Round

1. $4 \nabla 3 = (4 + 1)(3 - 1) = 5 \times 2 = \mathbf{10}$.

2. $x = x^3$ so $0 = x^3 - x = (x^2 - 1)x = (x + 1)(x - 1)x$, which has solutions -1, $+1$, and 0 for a total count of **3** integers.

3. Find x such that $256 = x + (x + 2) + (x + 4) + (x + 6) = 4x + 12$, so $4x = 244$. This results in $x = 61$, which is required to be odd, and we verify readily. Therefore, $x = \mathbf{61}$.

4. The area enclosed by square of side s is s^2; the perimeter of such a square is $4s$. Therefore, $s^2 = 5(4s) = 20s$ and $0 = s^2 - 20s = s(s - 20)$. Thus, s must be 0 or 20. A square with side 0 would be degenerate (just a point) and the question asks for the greatest possible value anyway, so we go with an answer of **20** units.

5. $f\left(g\left(h(10)\right)\right) = f\left(g(10 - 5)\right) = f\left(g(5)\right) = f(5^2 - 3) = f(22) = 22 + 2 = \mathbf{24}$.

6. At least 2 points are scored for each basket, which means at least 24 points for the 12 baskets made. Baskets might be worth 1 extra point each. Since a total of 29 points were scored, we need to account for $29 - 24 = 5$ extra points, which means **5** baskets worth 3 points.

7. The same number of chips stay in play throughout the game. There are 8 people (Alexandra and 7 others) who start, making $8 \times 60 = 480$ chips, and 3 players left after 5 players drop out. The average number of chips for each of the 3 remaining people is $480/3 = \mathbf{160}$ chips.

8. Regardless the probability of any particular value with the dart, the probability of the die rolling on that particular value is $\frac{1}{6}$. (To demonstrate this in case you do not believe it, let p_i be the probability of the dart hitting value i; the probability of the die rolling an i is $\frac{1}{6}$. Therefore, the probability for both the dart and die hitting i, as independent events, is the product $\frac{1}{6}p_i$. Add the products for all possible values of i and factor out the common $\frac{1}{6}$ to obtain $\frac{1}{6}(p_1 + p_2 + p_3 + p_4 + p_5 + p_6)$. The sum of the probabilities in the parentheses must be 1, since the dart hits some value, so the result is indeed $\mathbf{\frac{1}{6}}$.

9. $33 \times 34 \times 57 \times 65 = (3 \times 11) \times (2 \times 17) \times (3 \times 19) \times (5 \times 13)$ is the prime factorization. Each prime, except 3, occurs once, and 3 occurs twice to be the only perfect square, thus **3**.

10. The sum of the elements in each row, column, and diagonal of a 3×3 magic square must be $\frac{1+2+3+4+5+6+7+8+9}{3} = \frac{9\times10}{2\times3} = 15$. For the center column to sum to 15 the bottom box must be $10 - x$. Then for the bottom row to sum to 15, the left box must be 4. Then for the bottom-left to top-right diagonal to sum to 15, the top right box must be 6. Then for the right column to sum to 15, the middle right box must be $8 - x$. For the top row to sum to 15, the left box must be $9 - x$. Lastly, for the left column to sum to 15, the middle box must be $x + 2$. Now we see three sets of 3 consecutive values: x, $x + 1$, $x + 2$; 4, 5, 6; and $8 - x$, $9 - x$, $10 - x$. Either one of the unknown sets must evaluate to 1, 2, 3 and the other to 7, 8, 9; x is the least value in its set. Therefore, x can be either 1 or 7, the sum of which is **8**.

$9 - x$	x	6
$x + 2$	5	$8 - x$
4	$10 - x$	$x + 1$

11. The two sections of the rectangle are trapezoids. The line intersects the rectangle at the top where $y = 3$, so $x = 5 - 3 = 2$, and at the bottom where $y = 0$, so $x = 5 - 0 = 5$. The average of the base lengths of the left trapezoid is $(2 + 5)/2 = 7/2$, and the rectangle has width 8, so the average of the base lengths of the right trapezoid is $8 - \frac{7}{2} = 9/2$. Using the height to find the actual area of the trapezoids is unneeded since it will cancel out in the ratio, so only the average base lengths are needed. The smaller is 7/2 and the larger is 9/2, so the ratio of the smaller to the larger has the 2s in the denominator cancel, leaving $\mathbf{7/9}$.

12. Distance equals the product of rate and time: distance is 1 mi, rate is p mi/h, and time is p minutes. Therefore, $1\text{ mi} = p\ \frac{\text{mi}}{\text{h}} \times p\text{ min} = p^2\text{ mi} \cdot \frac{\text{min}}{\text{h}}$. Now, $1\text{ h} = 60\text{ min}$, so $\frac{\text{min}}{\text{h}} = \frac{1}{60}$. Therefore, $1\text{ mi} = (p^2/60)\text{ mi}$. The "mi" cancel, and we are left with $p^2 = 60$. Since $7^2 = 49$ and $8^2 = 64$, p must be between 7 and 8, and 60 looks closer to 64 than to 49, it appears 8 is the correct answer. We can double check with $7.5^2 = 7(7 + 1) + 0.25 = 56.25 < 60$, so, yes, the correct answer is **8**.

13. An arithmetic sequence involves a common difference, which may be either positive or negative, between consecutive terms.
Having a negative difference means the terms are decreasing, so the last two must satisfy $\mathtt{BC7} \geq \mathtt{B91}$, which necessitates $\mathtt{C} = 9$ and yields a common difference of −6. That means the first term must be 12 greater than the third term and the `B97 + 12` must end in `09`, but the first term ends in `99`. A contradiction results, so the common difference must be positive.
The difference in the last pair of values, $\mathtt{B91} - \mathtt{BC7}$ must be in the range 4 (for $\mathtt{C} = 8$) to 84 (for $\mathtt{C} = \mathtt{0}$) and end in 4. The difference between the last and first terms is 3 times the common difference, which is in the range 12 to 252, ending in 2 and divisible by 3; that difference is $\mathtt{B91} - \mathtt{C99}$, which ends in 92. The choices satisfying the range constraint are 92 and 192, of which only 192 is divisible by 3. Thus, the common difference is $192/3 = 64$, making the third term 64 less than the fourth term, that is, $\mathtt{B91} - 64 = \mathtt{B27} = \mathtt{BC7}$, so $\mathtt{C} = 2$. That makes the first term be 299 and successive terms being 363, 427, and 491, which is consistent with the given information, with $\mathtt{A} = 3$, $\mathtt{B} = 4$, and $\mathtt{C} = 2$. Therefore, $\mathtt{A}^2 + \mathtt{B}^2 + \mathtt{C}^2 = 9 + 16 + 4 = \mathbf{29}$.

14. $x \star y = x^2 + xy = x(x + y)$. Therefore, $9 = a \star b = a(a + b)$, and $72 = b \star a = b(b + a) = b(a + b)$. Now, $(a + b) \neq 0$ since the products are not 0, so we may divide by an expression containing $(a + b)$ as a factor. Dividing the upper equation by the lower equation yields:
$\frac{\mathbf{1}}{\mathbf{8}} = \frac{a(a + b)}{b(a + b)} = \frac{a}{b}$.

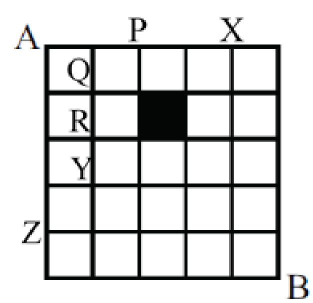
15. Let d be the distance to the contest and t be the time to travel to the contest. Then the average speed going to the contest is d/t, which I will call v (for the magnitude of the velocity). Returning home involves a distance 50 % greater than going, so $(1 + 0.5)d = 1.5d$ and a time twice as long as going, so $2t$, making the average return speed $\frac{1.5d}{2t} = \frac{1.5}{2}\frac{d}{t} = \frac{3}{4}v$, which is given to be 10 mi/h less than v. Therefore, $\frac{3}{4}v = v - 10\text{ mi/h}$ and $10\text{ mi/h} = v - \frac{3}{4}v = v/4$, so $v = 4 \times 10\text{ mi/h} = \mathbf{40}\text{ mi/h}$.

16. Let T represent the time of one orbit, called the orbital period, and apply a subscript F when applied to Flion, H when applied to Hathov, and R when applied to Reflurn. Then, we want $T_{\mathrm{H}}/T_{\mathrm{R}}$, so let's express T_{H} in terms of T_{F} and T_{F} in terms of T_{R}, since we are given the relationship between Flion and each of Hathov and Refurn:
$5T_{\mathrm{F}} = 2T_{\mathrm{R}}$, so $T_{\mathrm{F}} = \frac{2}{5}T_{\mathrm{R}}$, and $19T_{\mathrm{F}} = 3T_{\mathrm{H}}$, so $T_{\mathrm{H}} = \frac{19}{3}T_{\mathrm{F}} = \frac{19}{3}\left(\frac{2}{5}T_{\mathrm{R}}\right) = \frac{38}{15}T_{\mathrm{R}}$, so $T_{\mathrm{H}}/T_{\mathrm{R}} = \mathbf{\frac{38}{15}}$.

17. Three times they tripled the opponents' score, so their scores must be divisible by 3; there are three such scores (30, 33, 66), so they must be those scores, meaning the opponents' scores in those games were 10, 11, and 22, respectively. For three of the remaining five scores, the team doubled their opponents' scores, so their scores must be divisible by 2; there are three such scores (22, 44, 50), meaning the opponents' scores in those games were 11, 22, and 25, respectively. The two remaining scores (55 and 61) fell short of the opponents by 4, so the opponents' scores for those two games were 59 and 65. The sum of the eight scores of the opponents is, therefore, $10 + 11 + 22 + 11 + 22 + 25 + 59 + 65 = \mathbf{225}$ points.

18. For an integer to be divisible by 72, the three least significant digits taken together as an integer must be divisible by 8 and the sum of all the digits must be divisible by 9. The largest sum that one can obtain from 100 digits is when all digits are 9, yielding a sum of 900. However, the last three digits taken as an integer, 999, is not divisible by 8. Thus, we must decrease at least one of the digits below 9, making the sum below 900. The next largest potential value sum (requiring that sum to be divisible by 9) is 891. If we find a way to achieve a sum of 891 while having divisibility of the last three digits together by 8, then we are done. The greatest three-digit multiple of 8 is 992, which has 20 as the sum of its digits, yielding a decrease of 7, 2 short of what we need to decrease. That can be remedied by decreasing the thousands digit from 9 to 7. Constructing the 100-digit number as 96 copies of 9 followed by the last 4 digits being 7992 yields a value whose digits sum to 891, which is divisible by 9, and the last 3 digits as the integer 992 that is divisible by 8. Therefore, the greatest possible sum of the digits is **891**.

19. The point B is 5 segments to the right and 5 segments down from A, so the shortest possible path sequence involves 10 segments, which is the length specified for allowed sequences. In order to achieve such, backtracking (moving back to the left or back up) is impossible. Due to constraints on avoiding the shaded box, if a path goes through P it must proceed to X; similarly if a path goes through Q or R it must proceed to Y. That means every path must go through exactly one of X, Y, or Z. The number of paths going from A to B through X is the product of the number of paths from A to X times the number of paths from X to B, and similarly through Y and through Z. The number of available paths that involves moving d steps down (D) and r steps right (R) is the number of arrangements (combinations) of the Ds and Rs, given by ${}_{d+r}C_d = {}_{d+r}C_r$.



From A to X: $d = 0$ and $r = 4$; from X to B: $d = 5$ and $r = 1$. ${}_4C_0 = 1$ and ${}_6C_1 = 6$.
From A to Y: $d = 3$ and $r = 1$; from Y to B: $d = 2$ and $r = 4$. ${}_4C_1 = 4$ and ${}_6C_2 = 15$.
From A to Z: $d = 4$ and $r = 0$; from Z to B: $d = 1$ and $r = 5$. ${}_4C_0 = 1$ and ${}_6C_1 = 6$.
Therefore, the total number of valid paths results in $1 \times 6 + 4 \times 15 + 1 \times 6 = \mathbf{72}$ ways to get from A to B.

20. $|3x - 3| < 13$ is equivalent to $-13 < 3x - 3 < 13$, which is equivalent to $-\frac{13}{3} < x - 1 < \frac{13}{3}$. Therefore, $-\frac{10}{3} = -3\frac{1}{3} < x < \frac{16}{3} = 5\frac{1}{3}$. The integers satisfying those constraints are −3, −2, …, 3, 4, 5. When we add these, the −3 through −1 cancel out the 1 through 3, leaving only 4 and 5, so the answer is $4 + 5 = \mathbf{9}$.

21. The lattice points must satisfy $3x - 24 < y < 0$, equivalent to $3x - 23 \leq y \leq -1$ for x an integer from 1 through 7. For $x = 1$, y must be an integer from −1 through −20, for a total of 20 choices. As we increment x by 1, the minimum value of y increases by 3, decreasing the number of choices by 3. This continues through $x = 7$, where y must be one of two choices, −1 or −2. The total count of qualifying lattice points is the sum of an arithmetic series 20, 17, 14, 11, 8, 5, 2, which has a total of 7 terms (one for each of the 7 values of x). The sum of an arithmetic series is the product of the number of terms times the average of the first and last terms, thus there are $7 \times (20 + 2)/2 = 7 \times 11 = \mathbf{77}$ points.

22. A and B are in the fractional part of the value, so we need not concern ourselves with the integer part. The fractional part, based on the base 4 expression, is given by: $\frac{0}{4^1} + \frac{2}{4^2} + \frac{1}{4^3} = \frac{1}{8} + \frac{1}{64} = \frac{1}{8^1} + \frac{1}{8^2} = \frac{A}{8^1} + \frac{B}{8^2}$. Therefore, A = B = 1, so A + B = **2**.

23. This is a variant of the Monty Hall problem. When Bryan firsts picks a box, the probability of him picking the one with the coin is 1/10 and of not picking the one with the coin is 9/10. After 5 boxes are removed, there are 5 left. In the 1/10 case, Bryan had the good box and a change would have 0 probability of yielding the good box. In the 9/10 case, Bryan does not have the good box, and the coin is equally likely to be in any one of the other 4 boxes, thus probability 1/4 to change to the good box. Thus, the probability of changing to the good box is $\frac{1}{10} \times 0 + \frac{9}{10} \times \frac{1}{4} = \mathbf{\frac{9}{40}}$.

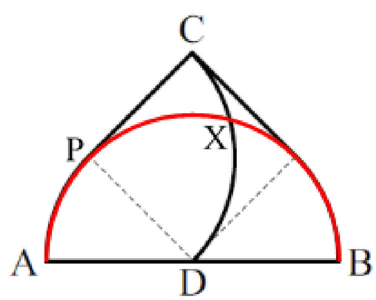
24. The three numbers can be represented as $10a + 7$, $10b + 7$, and $10c + 7$, where a, b, and c are integers in the range between 1 and 9, inclusive. When we add these numbers together we obtain: $10a + 10b + 10c + 21 = 10(a + b + c + 2) + 1$. The 1 tells us the units digit of the sum, and the $a + b + c + 2$ corresponds to the tens digit of the sum—however, $a + b + c + 2$ might exceed 9, in which case we need to take only the units digit of the latter sum.
Now, the product of the three given numbers is obtained as follows:
The product of the three 7s contributes 343. Taking each of the three terms containing the factor 10 in succession and multiplying by the 7s in the other two terms contributes $490a + 490b + 490c = 10\big(49(a + b + c)\big)$. The other contributions to the product will involve at least two factors of 10, yielding 100 times some integer, thus ending in 00 and not affecting the tens digit of the product. Thus, the product is $100x + 10\big(49(a + b + c)\big) + 343$. Let's reduce some clutter here. The last (rightmost) two digits of an integer can be obtained as a pair by dividing the integer by 100 to yield an integer quotient that we throw way and a remainder that is the result we want (modulo 100 arithmetic). The $100x$ goes to 0 and disappears, 490 goes to 90, so the second term goes to $10\big(9(a + b + c)\big)$ and 343 goes to 43. Thus, what we care about from the product is $10\big(9(a + b + c)\big) + 43 = 10(9(a + b + c) + 4) + 3$, so that the product has units digit 3 and tens digit given by the units digit of $9(a + b + c) + 4$. We are told this tens digit is 5; therefore, $9(a + b + c)$ ends in 1. For 9 times an integer to end in 1, the integer must end in 9. So, now we know $a + b + c$ ends in 9, but, remember, we need 2 more than that, which yields a desired digit value of **1**.

25. Selecting three real numbers x, y, and z, each in the range 0 to 1 (and it does not matter whether we include versus exclude 0, and likewise for 1) is like constructing a unit cube with one vertex at the origin, one edge along the x-axis out to 1, an adjacent edge along the y-axis out to 1, and another adjacent edge along the z-axis going out to 1. The criterion $x + y + z < 1$ corresponds to being at a point on the origin side of the plane $x + y + z = 1$, which plane cuts through the unit cube. The region in question is summarized as: $x > 0$; $y > 0$; $z > 0$; $x + y + z < 1$, describing the interior of a tetrahedron with vertices (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1). More specifically this is a pyramid whose base is an isosceles right triangle with legs of length 1, and the pyramid height is 1. The volume of a pyramid is 1/3 times the base area. The base area is $(1/2) \times 1 \times 1 = 1/2$, so the volume is $(1/3)(1/2)(1) = 1/6$. The volume of the unit cube, which is $1^3 = 1$ is the volume of the whole sample space. Therefore, the probability of a random position in the cube being within the desired pyramid is $\frac{1}{6}/1 = \mathbf{\frac{1}{6}}$.

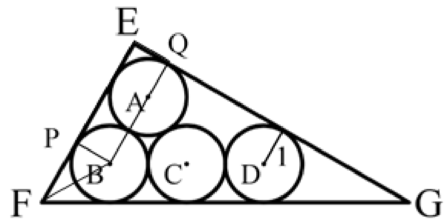
26. The pattern goes in pairs. The first value of each pair is n^2 and the second value is $n^2 + n = n(n+1)$, where n is the pair number starting with 1. (To get the next value, add $n + 1$ to obtain $(n+1)^2$, which is indeed the first element of the next pair.) The first time to add 2000 is going from the second value of pair 1999 to the first value of pair 2000. Thus, the element to which the first time 2000 is added is $1999(2000) = \mathbf{3{,}998{,}000}$.

27. The total number of possible outcomes is $6^3 = 216$. For the desired outcomes, we have a variety of cases to deal with. All three dice show the same value: (3, 3, 3), with 1 way for that to occur. Two dice show the same value and the third die shows a different value: (1, 1, 5); (2, 2, 4); (3, 3, 1); (3, 3, 2); (3, 3, 4); (3, 3, 5); (3, 3, 6); (4, 4, 2); (5, 5, 1), each with 3 ways to occur (which of the three dice is the different one). All three dice have different values: (1, 5, 2); (1, 5, 3); (1, 5, 4); (1, 5, 6); (2, 4, 1); (2, 4, 3); (2, 4, 5); (2, 4, 6), each with 6 ways to occur. This yields a total count of desired outcomes is $1 \times 1 + 9 \times 3 + 8 \times 6 = 76$. The probability of the ratio of count of desired outcomes to count of total outcomes: $\frac{76}{216} = \mathbf{\frac{19}{54}}$.

28. Let P be the leftmost vertex of the unit square. Then arc $\widehat{AP}$ is a circular arc of measure 45° centered at D. At D the triangle ADB would be so obtuse that it is degenerate. Moving up the arc toward C makes the triangle less and less obtuse. At some point on the arc the triangle becomes right, and above that toward C the triangle will be acute. Thus, we need to find the point X on $\widehat{CD}$ such that ∠AXB is right and determine the ratio of the measure of arc $\widehat{CX}$ to the measure of the arc $\widehat{CD}$. Now, segment AB is a diameter of circle D, and a triangle inscribed in a circle such that one side of the triangle is a diameter of the circle has a right angle at the vertex opposite the diameter; arc $\widehat{AP}$ is a portion of circle D, and the upper semicircle of D containing points A and B is colored red. Therefore, X must be on circle D; from the information we are given, we know that arc $\widehat{CD}$ is part of a circle centered at P. Therefore, X is at an intersection of circles D and P. Now segments PX and PD are radii of circle P, and segments DP and DX are radii of circle D. Therefore, segments DX, XP, and PD are congruent and form an equilateral triangle. Therefore, ∠DPX and arc $\widehat{XD}$ have measure 60°. ∠DPC (a right angle) and arc $\widehat{CD}$ have measure 90°. Thus, the measure of arc $\widehat{CX}$ is $90° - 60° = 30°$, and the fraction of the arc forming an acute triangle is $30°/90° = \frac{1}{3} = 33\frac{1}{3}$ %, which rounds to the nearest tenth of a percent as **33.3** percent.



29. The segments AB, BC, and CA form an equilateral triangle, so $\angle ABC$ has measure $60°$. Segment AB is parallel to segment EF and segment BC is parallel to segment FG, so $\angle EFG$ is congruent to $\angle ABC$, which has measure $60°$. Because C in on segment BD, $\angle ACD$ is supplementary to $\angle ACB$, so $\angle ACD$ has measure $180° - 60° = 120°$. Segments CA and CD are congruent, so triangle ACD is isosceles and base angle $\angle ADC$ has measure that is half the supplement of $\angle ACD$, thus $60°/2 = 30°$. Segment BD is parallel to segment FG and segment AD is parallel to segment EG, so $\angle ADC$ and $\angle EGF$ are congruent with measure $30°$. Therefore, triangle GFE is a 30°-60°-90° triangle and the length of hypotenuse FG is double the length of short leg FE. The length of FE is the sum of the lengths of FP and PE. Segments PE and BQ are congruent, with the length of the latter being the diameter of circle A plus the radius of circle B, thus a total of 3 meters. Triangle FBP is a 30°-60°-90° triangle, so long leg FP is $\sqrt{3}$ times as long as short leg BP, which is $\sqrt{3} \times 1$ meter $= \sqrt{3}$ meters. Therefore, segment FE has length $\left(3 + \sqrt{3}\right)$ meters, and FG is double that for $\left(6 + 2\sqrt{3}\right)$ meters. The problem requests $6 + 2 + 3 = \mathbf{11}$.



30. Let's set up a 3-dimensional coordinate system to ease our calculations. A plane can be expressed in the form $ax + by + cz = d$. If we pick the origin to be a point the plane passes through, the calculations are slightly simplified because in such a setup, $d = 0$. The plane goes through the vertex A, and a vertex is often a great place to put the origin, so let's choose $A = (0; 0; 0)$. Let's pick B along the x-axis, and it is given as distance 2 from A, so $B = (2; 0; 0)$. Let's pick D along the y-axis, and it is given as distance 2 from A, so $D = (0; 2; 0)$. Then $C = (2; 2; 0)$. The figure is stated to be a right pyramid: the centroid of the square base is then at the midpoint of segment AC, which is also the midpoint of segment BD, so $(1; 1; 0)$; P is distance 1 up, so at $(1; 1; 1)$.
B′ is 1/4 of the way from B to P, so $B' = (2;\ 0;\ 0) + \frac{1}{4}[(1;\ 1;\ 1) - (2;\ 0;\ 0)] = \left(\frac{7}{4};\ \frac{1}{4};\ \frac{1}{4}\right)$.
D′ is 1/5 of the way from D to P, so $D' = (0;\ 2;\ 0) + \frac{1}{5}[(1;\ 1;\ 1) - (0;\ 2;\ 0)] = \left(\frac{1}{5};\ \frac{9}{5};\ \frac{1}{5}\right)$.
The equation of the given plane must be of the form $ax + by + cz = 0$ for A to be on the plane. However, multiplying the equation through by nonzero k yields an equally valid solution, so there are only two independent parameters among a, b, and c. Because the given plane is clearly not parallel to any of the coordinate planes (xy, xz, yz), we may choose any the three coefficients to be fixed at 1, so let's choose $c = 1$. Thus, the given plane must satisfy the equation $ax + by + z = 0$ for some a and b. For B′ to be on the plane, we must satisfy: $\frac{7}{4}a + \frac{1}{4}b + \frac{1}{4} = 0$, which can be simplified to $7a + b = -1$. Similarly, for D′ to be on the plane, we must satisfy: $\frac{1}{5}a + \frac{9}{5}b + \frac{1}{5} = 0$, which can be simplified to $a + 9b = -1$. Multiplying the first of these equations by 9 and subtracting the second equation yields: $62a = -8$, so $a = -\frac{4}{31}$; $b = -1 - 7a = -1 + \frac{28}{31} = -\frac{3}{31}$. The point C′ to be on segment PC must have a coordinate of the form $(2 - z;\ 2 - z;\ z)$ for some y satisfying $0 < z < 1$. (This has been set up for z to be the answer to the problem, as the fraction of the distance from C to P as where C′ is.) Therefore, $0 = -\frac{4}{31}(2 - z) - \frac{3}{31}(2 - z) + z = -\frac{14}{31} + \frac{38}{31}z$, so $z = \frac{14}{38} = \frac{\mathbf{7}}{\mathbf{19}}$.

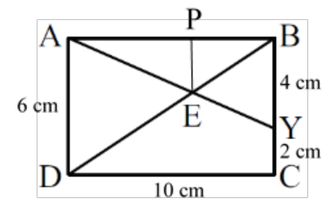
2018 State Competition Target Round

1. Company 1: $\$500 + 5000\text{ km} \times \$0.30\ \frac{1}{\text{km}} + 500\text{ kg} \times \$0.40\ \frac{1}{\text{kg}} = \$500 + \$1500 + \$200 = \$2200$.
   Company 2: $\$0 + 5000\text{ km} \times \$0.50\ \frac{1}{\text{km}} + 500\text{ kg} \times \$0.30\ \frac{1}{\text{kg}} = \$2500 + \$150 = \$2650$.
   Subtracting the greater amount minus the lesser amount yields $\$2650 - \$2200 = \$\mathbf{450}$.

2. The count 10 dozen means $10 \times 12 = 120$ donuts. There are $\binom{120}{5}$ combinations of donuts sampled, which is $\frac{120!}{5!\,115!}$. For acceptance, 0 of the 4 bad ones and 5 of the 116 good ones must be in a sample, allowing for a total number of combinations of $\binom{4}{0}\binom{116}{5} = \frac{4!}{0!\,4!} \times \frac{116!}{5!\,111!} = \frac{116!}{5!\,111!} \Big/ \frac{120!}{5!\,115!} = \frac{116\times115\times114\times113\times112}{120\times119\times118\times117\times116} = \frac{115\times114\times113\times112}{120\times119\times118\times117} \approx 0.8416$, which rounds to 84%.
   Therefore, the probability of rejection is 100% minus this value, or **16**%.

3. There is a gain of $\$8 - \$3 = \$5$ for each bigon sold. With \$400 upfront costs to make up for at \$5 per bigon, to break even, she must sell $\$400/\$5 = \mathbf{80}$ bigons.

4. Draw PE perpendicular to AP, as shown. Then triangle APE is a right triangle, and it is the hypotenuse AE that we wish to determine. Triangles AED and YEB are similar (Angle-Angle-Angle) with a scaling ratio of 6 cm to 4 cm, or 3:2. Therefore, the distance of E from the left side of the rectangle is $\frac{3}{5}(10\text{ cm}) = 6\text{ cm}$, which is also the length of AP.



   Triangles APE and ABY are similar (Angle-Angle-Angle) with a scaling ratio of 6 cm to 10 cm, or 3:5. Therefore the length of PE is 3/5 of the length of BY, so $\frac{3}{5}(4\text{ cm}) = \frac{12}{5}$ cm. We can now apply the Pythagorean Theorem to legs of $\frac{12}{5}$ cm and 6 cm $= \frac{30}{5}$ cm. We have
   $\sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{30}{5}\right)^2}\text{ cm} = \sqrt{\frac{1044}{25}}\text{ cm} = \boldsymbol{\frac{6\sqrt{29}}{5}}$ cm.

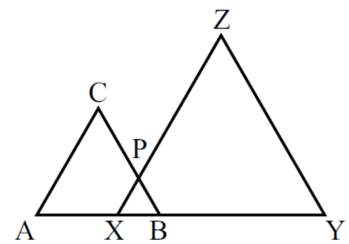
5. We need to add all the individual scores and divide by the total count of scores. Many scores are held by multiple competitors, and the histogram tells us how many individuals earned each score value, so we must multiply each score value by the corresponding number of students to obtain the sum of all the individual scores. Therefore, the mean score is:

$$\frac{2\times0+4\times2+11\times4+13\times6+12\times8+7\times10+0\times12+1\times14}{2+4+11+13+12+7+0+1}$$

$$=\frac{0+8+44+78+96++70+0+14}{50}=\frac{310}{50}=\mathbf{6.2}.$$

6. Whenever a line of slope 1 bisects an angle formed by two lines intersecting, the product of the slopes of the two lines is 1, except when one is horizontal and the other vertical—in other words, the two slopes are reciprocals of each other. Therefore, $n = 1/m$, so:
$2\sqrt{65} = m + n = m + \frac{1}{m}$, and $m^2 - 2\sqrt{65}m + 1 = 0$. Now, since the constant coefficient divided by the leading coefficient is $1/1 = 1$, the product of the two roots of this equation is 1, so that they are reciprocal to each other. Thus, the solution to this quadratic equation is two roots, one of which (the greater) is m and the other (the lesser) is n. The roots to the quadratic equation are: $\frac{2\sqrt{65}\pm\sqrt{260-4}}{2} = \sqrt{65} \pm 8$, with m being the greater, $\sqrt{65}+8$, and n being the lesser, $\sqrt{65} - 8$. Subtracting the two yields $m - n = (\sqrt{65} + 8) - (\sqrt{65} - 8) = 8 + 8 = \mathbf{16}$.

7. Triangles ABC and XYZ are given to be equilateral, so $m\angle PXB = m\angle ZXY =$ 60°, and $m\angle PBX = m\angle CBA =$ 60°, making triangle XBP equilateral. The perimeter of the pentagon AYZPC is the perimeter of triangle ABC plus the perimeter of triangle XYZ minus the perimeter of triangle XPB (the last "minus" part to take away the inappropriate inclusion of XP and PB, as well as to correct for double counting XB. Therefore, the requested perimeter is 3×3 cm $+ 3 \times 5$ cm $- 3 \times 1$ cm $=$ **21** cm.



8. Let's let p be the number of pages in each book and f be the number of pages Varun read to finish the first book, with $21 \le p \le 999$ and $1 \le f \le 41$; that leaves $42 - f$ pages to start the second book.
Therefore, Varun reads pages $p - f + 1$ through p from the first book and pages 1 through $42 - f$ from the second book. This makes two arithmetic series to evaluate and the results together: the sum of an arithmetic series is the product of the number of terms times the average of the first and last values in the series. The total for the two series is 2018. Therefore, $2018 = f \cdot \left(\frac{p-f+1+p}{2}\right) + (42 - f) \cdot \left(\frac{1+42-f}{2}\right)$. Multiplying through both sides by 2 yields:
$4036 = f \cdot (2p - f + 1) + 42(43 - f) - f(43 - f) = f \cdot (2p - f + 1 - 42 - 43 + f) + 1806$.
Thus, $2230 = f \cdot (2p - 84)$ and $1115 = f \cdot (p - 42)$, which means that f must be a divisor of $1115 = 5 \times 223$. (Note 223 is prime because it is not divisible by any of 2, 3, 5, 7, 11, or 13.) Thus, f must be one of 1, 5, 223, or 1115, but the last two possibilities are rejected because $f < 42$; also $p = \frac{1115}{f} + 42$, which must be less than 1000, so f cannot be 1. The only possibility is $f = 5$, which means $p = \frac{1115}{5} + 42 = 223 + 42 = \mathbf{265}$.

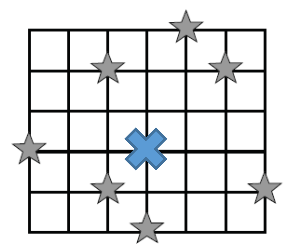
2018 State Competition Team Round

1. The sum of the squares of the first ten positive integers is 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 = 385. The arithmetic mean of these ten squares is 385/10 = **38.5**. If you know that $n(n+1)(2n+1)/6$ is the formula for the sum of the squares of the first n positive integers, you may note that finding the average of those squares requires diving that sum by n to obtain $(n+1)(2n+1)/6$. In this case, $n = 10$, so we have $\frac{(10+1)(20+1)}{6} = 11 \times \frac{7}{2} = \mathbf{38.5}$.

2. The name MATHCOUNTS has 7 constants (MTHCNTS—counting all occurrences, including repetitions) and 3 vowels (AOU). The probability of choosing a consonant for the first letter is 7/10, leaving 6 consonants and 3 vowels for choosing the second letter. That makes the probability of choosing a consonant for the second letter $6/9 = 2/3$. The probability of two independent event occurring is the product of the two probabilities for the individual events: $\frac{7}{10} \times \frac{2}{3} = \frac{14}{30} = \mathbf{\frac{7}{15}}$.

3. Since $(n+2)^2 - (n-2)^2 = (n^2+4n+4) - (n^2-4n+4) = 8n, (m+1)^2 - (m-1)^2 = (m^2+2m+1) - (m^2-2m+1) = 4m$, and we are told that these two expressions are equal, we have $8n = 4m$. Dividing both sides by 4 yields $2n = m = an$, which means $a = \mathbf{2}$.

4. When one wishes to minimize the sum of distances in along one dimension, the optimum position is at the median position. Here we have two dimensions, with each working independently of the other. There are 7 restaurants, so the fourth acts as the median. The blue cross marks the optimum position—the same horizontal coordinate as the fourth restaurant from left to right and the same vertical coordinate as the fourth restaurant from top to bottom. The cumulative horizontal distances as a number of kilometers is $3+1+1+0+1+2+3 = 11$; the cumulative vertical distances as a number of kilometers is $3+2+2+0+1+1+2 = 11$. The total distance of travel to go separately from the optimum point to each restaurant, is the sum of these two values, $11 \text{ km} + 11 \text{ km} = 22$ km. The average distance to each restaurant is this sum divided by 7, thus $\mathbf{\frac{22}{7}}$ km.

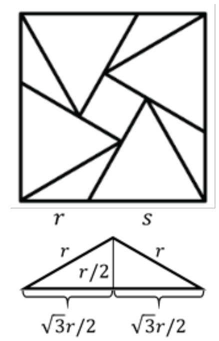


5. Profit is the difference of the selling price minus the purchase price. The percentage profit is 100 percent times the ratio of this difference to the purchase price. The total selling price is $2 \times \$25{,}000 = \$50{,}000$. The given profit rates cannot be exact, because they would involve purchase prices involving fractions of cents, so we will round each to the nearest cent. The purchase price of car 1 is $\$25{,}000/(1+0.22) = \$20{,}491.80$, and the purchase price of car 2 is $\$25{,}000/(1-0.07) = \$26{,}881.72$, for a total purchase price of \$47,373.52. The profit is $\$50{,}000.00 - \$47{,}373.52 = \$2{,}626.48$, for a percentage profit of: $100\% \times (\$2{,}626.48/\$47{,}373.52) = 5.54419\ldots$ percent. Which rounds to **5.54** percent.

6. A sequence of n 6s forming the digits of an integer is equal to $\frac{2}{3}10^n - \frac{2}{3}$. Thus, the numerator and the denominator of the 5 given fractions are $\left(A + \frac{2}{3}\right)10^n - \frac{2}{3}$ and $\left(6 + \frac{2}{3}\right)10^n - \frac{2}{3} + B - 6$, respectively, where n is an integer from 0 through 4 (and it will work for larger values of n as well). Because this ratio is the same for all n, the ratio of the two coefficients of 10^n must be the

same as the ratio of the two constant terms, so that $\frac{A+\frac{2}{3}}{20/3} = \frac{2/3}{\frac{20}{3}-B}$. Multiplying through numerator and denominator by 3 simplifies this to $\frac{3A+2}{20} = \frac{2}{20-3B}$. Flipping the two ratios yields $\frac{20}{3A+2} = \frac{20-3B}{2}$, so $20 - 3B = \frac{40}{3A+2}$, $3B = 20 - \frac{40}{3A+2} = \frac{60A+40-40}{3A+2} = \frac{60A}{3A+2}$. Therefore, $= \frac{20A}{3A+2}$. Both A and B are specified to be 1-digit integers, and B must not be 0 in order to avoid division by 0 in the fraction A/B, and that means A cannot be 0 either. If we try the integers 1 through 9 for A, we find an integer value results for B only for A being 1, 2, and 6, where B is 4, 5, and 6, respectively. The sum of these three values of A is $1 + 2 + 6 =$ **9**.

7. Let s be the desired length of a side of one of the equilateral triangles and r be the rest of the length 1 of a side of the unit square. Therefore, $r + s = 1$. The obtuse angle of the obtuse triangle is supplementary to the 60° angle of the equilateral triangle, thus 120°. The acute angle of the obtuse triangle at a vertex of the unit square is complementary to the 60° angle of the equilateral triangle, thus 30°. Therefore, the angles of the obtuse triangle measure 30°, 30°, and 120°. The base of the isosceles triangle matches a side s of the equilateral triangle. We can bisect the obtuse angle to form two congruent right (30-60-90) triangles, as shown, with hypotenuse r, short leg $r/2$, and long leg $\sqrt{3}r/2$. The base of the obtuse triangle is twice the length of this long leg, thus $\sqrt{3}r$. (Knowing and using the law of cosines can make this go faster.) Therefore, $s = \sqrt{3}r$, $r = s/\sqrt{3}$, and $1 = s + r = \left(1 + \frac{1}{\sqrt{3}}\right)s = \frac{\sqrt{3}+1}{\sqrt{3}}s$, making $s = \frac{\sqrt{3}}{\sqrt{3}+1} = \frac{\sqrt{3}}{\sqrt{3}+1}\frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{3-\sqrt{3}}{2} \approx$ **0.634** units.



8. There are $\frac{25!}{4!\,21!} = \frac{25\times\cancel{24\times}23\times22}{\cancel{4\times3\times2}} = 12{,}650$ sets of 4-card draws out of 25 cards. There are two cases satisfying the criteria:
(1) Three cards of the same number, each with different color; the fourth card must be a different number to have two numbers involved, and it must the same color as one of the first three cards to keep a total of three colors. Any of the 5 numbers may be the triplet number and they may have any 3 of 5 colors, so $\frac{5!}{3!\,2!} = 10$ color arrangements, for a total of $5 \times 10 = 50$ possible sets of triplet numbers. For each such set, there are 4 numbers and 3 colors that may be used to achieve the specified criteria, thus $50 \times 4 \times 3 = 600$ arrangements with a triplet number and three colors.
(2) Two cards of one number and two cards of another number. For four cards to have three colors, two of the cards must be the same color and they must have different numbers. The options for the two numbers out of 5 choices are $\frac{5!}{2!\,3!} = 10$ and likewise, independently, 10 for the colors of the first two card, so $10 \times 10 = 100$ options for the first two cards. The two cards of the second number have 2 choices for one card matching one of the already chosen colors and 3 choices for the other card to match one of the other 3 colors, making $100 \times 2 \times 3 = 600$ arrangements with two doublet numbers and three colors.
Therefore, there are a total of $600 + 600 = 1200$ desirable sets of 4 cards out of a possible 12 650 possible sets of 4 cards, which makes a probability of $\frac{1200}{12{,}650} = \mathbf{\frac{24}{253}}$.

9. The volume of a frustum is given by $\frac{h}{3}\left(B+\sqrt{Bb}+b\right)$, where h is the height of the frustum and B and b are the areas of the two end bases. The bottom base is a square of side 10 m, so area $(10\text{ m})^2 = 100\text{ m}^2$ and the top base is a square whose diagonal is 10 m, so area of half the square of the diagonal for 50 m^2. The height $h = 60$ m. Thus, the volume of the frustum is: $20\text{ m}\left(100\text{ m}^2+\sqrt{100\text{ m}^2\times 50\text{ m}^2}+50\text{ m}^2\right) = 3000\text{ m}^3 + 1000\sqrt{2}\text{ m}^3$. The volume of the pyramid is given by $\frac{H}{3}b$, where H is the height of the pyramid (not the same as h for the frustum), and b is the area of the base, which is the same as for the top base of the frustum, $10\text{ m}\times 50\text{ m}^2/3 = \frac{500}{3}\text{ m}^3$. The sum of the two parts is: $\left(3000+1000\sqrt{2}+\frac{500}{3}\right)\text{ m}^3 = 4580.88\ldots\text{ m}^3$, which rounds to the requested nearest 10 m^3 as **4580** m^3.

10. There are three cases:
$a_4 = 4^2 - a_1 - a_2 - a_3$;
$a_5 = 5^2 - a_2 - a_3 - a_4 = 5^2 - 4^2 + a_1$;
$a_6 = 6^2 - a_3 - a_4 - a_5 = 6^2 - 5^2 + a_2$;
$a_7 = 7^2 - a_4 - a_5 - a_6 = 7^2 - 6^2 + a_3$;
$a_8 = 8^2 - a_5 - a_6 - a_7 = 8^2 - 7^2 + 4^2 - a_1 - a_2 - a_3 = 8^2 - 7^2 + a_4$;
$a_9 = 9^2 - a_6 - a_7 - a_8 = 9^2 - 8^2 + 5^2 - 4^2 + a_1 = 9^2 - 8^2 + a_5$;
We have a pattern going for $k \geq 5$: $a_k = k^2 - (k-1)^2 + a_{k-4}$. We want to know about a_{99}.
$$\begin{aligned} a_{99} &= 99^2 - 98^2 + a_{95} = 99^2 - 98^2 + 95^2 - 94^2 + a_{91} = \cdots \\ &= (99^2-98^2)+(95^2-94^2)+(91^2-90^2)+\cdots+(7^2-6^2)+a_3 \\ &= 197+189+181+\cdots+13+a_3 \\ &= 24\times\frac{(197+13)}{2} + a_3 = 2520 + a_3. \end{aligned}$$
Going from the second line to the third line is based on $k^2-(k-1)^2 = 2k-1$; going from the third line to the fourth line is based on recognizing all but the last term on the third line being an arithmetic series of 24 terms with common difference 8.
To maximize this expression, all that is left to do is to maximize a_3. The only initializations $a_1;\ a_2;\ a_3$ that satisfy the criteria that all terms are positive integers and the terms are in strictly increasing order are:
1; 2; 3 yields 4th term 10 (but fails with 5th term 10);
1; 2; 4 yields 4th term 9;
1; 2; 5 yields 4th term 8;
1; 2; 6 yields 4th term 7;
1; 3; 4 yields 4th term 8;
1; 3; 5 yields 4th term 7;
1; 4; 5 yields 4th term 6;
2; 3; 4 yields 4th term 7;
2; 3; 5 yields 4th term 6.
The maximum value for a_3 is 6. Therefore, the maximum value for a_{99} is $2520 + 6 =$ **2526**.