MATHCOUNTS®

2018 Chapter Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2018 MATHCOUNTS® Chapter Competition. These solutions provide creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions with the MATHCOUNTS community.

2018 Chapter Competition Sprint Round

1. The next integer down from 2017.7012 is 2017, which is 0.7012 below; the next integer up is 2018, which is 0.2988 above. The latter is the lesser difference, so **2018** is the closest.

2. The minute hand (the longer hand) is pointing directly at the 7. Each increment of a number on the face corresponds to 5 minutes for the minute hand. Therefore, the clock is 7×5 minutes = **35** minutes after the hour.

3. Start with \$835;
   subtract \$415 for rent,
   leaving \$420;
   subtract \$220 for utilities,
   leaving \$**200**.

4. The halfway point occurs at the average of the two endpoints, which are at 0 and 8, so $p = (0 + 8)/2 = \mathbf{4}$.

5. $2 \times \$14$ and $14 \times \$2$ both equal $\$2 \times 14 = \$28$. Since we have this cost twice, the total is $2 \times \$28 = \$\mathbf{56}$.

6. $n + 4 = 10 - 2 = 8$. Therefore, $n = 8 - 4 = \mathbf{4}$.

7. a for apples and p for pears: $a/p = 3/5 = a/20$, so $a = 20 \times 3/5 = 4 \times 3 = \mathbf{12}$ apples.

8. $462 = 7 \times 11 \times p$, so $p = (462/11)/7 = 42/7 = \mathbf{6}$.

9. $1/0.25 - 2 = 1/(1/4) - 2 = 4 - 2 = \mathbf{2}$.

10. There are 5 values, so the arithmetic mean is the sum of all the values then divided by 5:
    $10 = (9 + 11 + 13 + x + 7)/5 = (40 + x)/5$, so $40 + x = 5 \times 10 = 50$.
    Therefore, $x = 50 - 40 = \mathbf{10}$.

11. $3(p + 4) + 4(p + 11) = 7(p + q)$;
    $3p + 12 + 4p + 44 = 7p + 7q$;
    $7p + 56 = 7p + 7q$;
    $56 = 7q$;
    $q = \mathbf{8}$.

12. $V = hwd = 10 \text{ inches} \times 8 \text{ inches} \times 3 \text{ inches} = (10 \times 8 \times 3) = \mathbf{240}\ \text{in}^3$.

13. $\frac{z}{16} = \frac{4}{z}$. Multiply both sides by $16z$:
    $z^2 = 64$, so $z = \pm 8$.
    We are directed to take the positive value, so $z = \mathbf{8}$.

14. She has $8 - 1 = 7$ non-rotten apples costing \$3.50.
Therefore, each non-rotten apple cost $\$3.50/7 = \$0.50 =$ **50** cents.

15. $123{,}456 + 1980 = 125{,}436$, which has swapped the 3 and the 5.
$3 \times 5 =$ **15**.

16. The sum of the measures of the three angles of a triangle is always 180 degrees. Two of the angles are 35 degrees and 95 degrees, totaling 130 degrees, so the remaining angle must have measure 180 degrees – 130 degrees = **50** degrees.

17. The values of the 25¢, 10¢, 5¢ and 1¢ are separated enough that duplicated sums cannot occur. In any possible mix of the coins, the quarter can be included or excluded (2 options), likewise for the dime (2 options), the nickel (2 options), and the penny (2 options). Each of these 4 inclusion-exclusion cases is independent of the others, so there are $2 \times 2 \times 2 \times 2 = 16$ possible mixes of coins; however, one of these, excluding all 4 coins, does not meet the criterion that at least one coin must be used. Therefore, there are **15** possible amounts.

18. $x + y = 14$ and $x - y = 4$.
The sum of these two equations is $2x = 18$, so $x = 9$.
Thus, $9 + y = 14$, so $y = 5$.
Therefore, $xy = 9 \times 5 = \mathbf{45}$.

19. $x^2 - y^2 = (x + y)(x - y)$. Let $x = 16$ and $y = 15$.
Therefore, $16^2 - 15^2 = (16 + 15)(16 - 15) = 31 \times 1 = \mathbf{31}$.

20. The mean of x and y is $\frac{x+y}{2}$. Therefore,
$\frac{a+b}{2} = 6.2$;
$\frac{b+c}{2} = 7.3$;
$\frac{a+c}{2} = 4.5$.
We see that each of $a/2$, $b/2$ and $c/2$ occurs twice in these three equations, so if we add the three equations together, we will get the desired $a + b + c$.
Therefore, $a + b + c = 6.2 + 7.3 + 4.5 = \mathbf{18}$.

21. If you know the first few powers of 3, this question is very easy and fast. For those who do not know that many powers of 3, we have $3^2 = 9$; $9^2 = 81$ and $9^2 = (3^2)^2 = 3^4$, so $3^4 = 81$. We are still below 100 but not by much. The next power of 3 is 243, which is too big.
Therefore, the largest integer exponent is **4**.

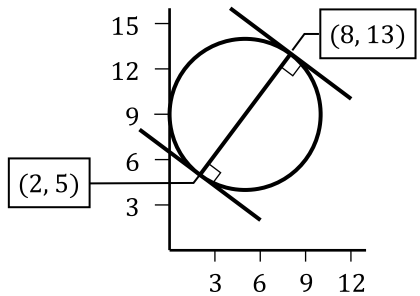
22. A number is divisible by 6 if and only if it is divisible by 2 and by 3. To be divisible by 2, the units digit must be even, so A must be 0, 2, 4, 6 or 8. To be divisible by 3, the sum of all the digits in the number must be divisible by 3, so $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + \text{A} = 44 + \text{A}$ needs to be 45, 48 or 51 to be divisible by 3 and have A be a digit. That means A must be 1, 4 or 7. The only value that satisfies both divisibility requirements is **4**.

23. We can make rectangles 1 column wide, in which case we have 3 choices, 2 columns wide, in which case we have 2 choices (#1 with #2 or #2 with #3), or all 3 columns wide, with only 1 choice. Thus, we have $3 + 2 + 1 = 6$ choices for arranging columns to make rectangles. The 3 rows work the same way as the 3 columns, so there are 6 choices for arranging rows. The manipulation of rows is independent of the manipulation of columns (for each of the 6 row choices, we have 6 column choices), making a total of $6 \times 6 =$ **36** rectangles.
More generally, if you have r rows, there will be T_r choices of row arrangements, where T_r is the rth triangular number, $r(r+1)/2$. Likewise, with c columns, there will be T_c choices of column arrangements, for a total of T_rT_c rectangles. In this problem, $T_3T_3 = 6 \times 6$.

24. The denominator of each of the 10 factors in $\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{9}{10}\cdot\frac{11}{12}\cdot\frac{13}{14}\cdot\frac{15}{16}\cdot\frac{17}{18}\cdot\frac{19}{20}$ is even and, therefore, has a factor of 2. Let's pull out the 10 factors of 2 in the denominators to yield: $\frac{1}{2^{10}}\cdot\frac{1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot13\cdot15\cdot17\cdot19}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}$, with both the numerator and denominator having factors 1, 3, 5, 7, and 9 that can be cancelled out, leaving $\frac{1}{2^{10}}\cdot\frac{11\cdot13\cdot15\cdot17\cdot19}{2\cdot4\cdot6\cdot8\cdot10}$. Again each of the 5 factors in the denominator of the fraction on the right is even and has a factor of 2 that can be pulled out, so let's remove them from the right fraction and give them to the denominator or the left fraction, now making 15 factors of 2 there: $\frac{1}{2^{15}}\cdot\frac{11\cdot13\cdot15\cdot17\cdot19}{1\cdot2\cdot3\cdot4\cdot5}$. Now, the factors of 3 and 5 together in the denominator on the right cancel with the 15 in the numerator, leaving $\frac{1}{2^{15}}\cdot\frac{11\cdot13\cdot17\cdot19}{1\cdot2\cdot4}$. The denominator on the right is now $8 = 2^3$, so we have 3 more factors of 2 to move to the 15 already in the denominator on the left, yielding $\frac{1}{2^{18}}\cdot 11\cdot13\cdot17\cdot19 = \frac{11\cdot13\cdot17\cdot19}{2^{18}}$. Now we have some arithmetic to do and this is Sprint Round—only the biological calculator above your shoulders, with assistance of pencil and paper, allowed. There are several ways to do this efficiently, and different people may well have different preferences. One way is to rearrange:
$11\cdot13\cdot17\cdot19 = (13\cdot17)(11\cdot19) = [(15+2)(15-2)][(15+4)(15-4)]$
$= (15^2-2^2)(15^2-4^2) = (15^2)^2 - (4+16)15^2 + (4\times16)$
$= 15^2(15^2-20) + 64 = 225\times205 + 64 = (215+10)(215-10) + 64$
$= 215^2 - 10^2 + 64.$
Now we need to square an integer 215 ending in 5. Break off that 5 and call the rest of the number n, so here $n = 21$. The square of such a number can be calculated is the number of 100s being $n(n+1)$ and the attaching 25 for the tens and units digits:
$21(21+1) = 21\times22 = 21\times2\times11 = 42\times11 = 462$, using the multiply by 11 trick.
Therefore, $215^2 = 46\,225$, so we need $46{,}225 - 100 + 64 = 46{,}125 + 64 = 46{,}189$, to which we need to add the exponent 18 that 2 was raised to: $46{,}189 + 18 =$ **46,207**.

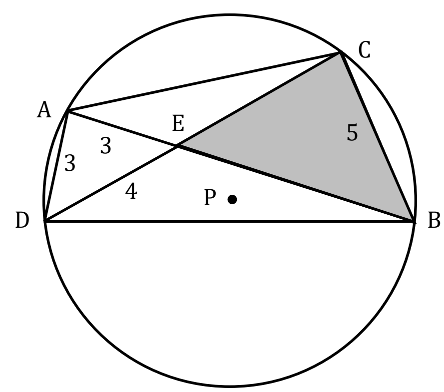
25. Let t be the total bonus amount; A be Arman's portion, B be Bernardo's portion, and C be Carson's portion.
$A = \frac{1}{3}t + \$10.$
$B = \frac{1}{2}(t-A) + \$3 = \frac{1}{2}\left[t - \left(\frac{1}{3}t + \$10\right)\right] + \$3 = \frac{1}{3}t - \$2.$
$C = \$25.$
$t = A + B + C = \frac{1}{3}t + \$10 + \frac{1}{3}t - \$2 + \$25 = \frac{2}{3}t + \$33,$
so $\frac{1}{3}t = \$33$ and $t = \$$**99**.

26. A tangent line to a circle is always perpendicular to the radius connecting the center of the circle to the point of tangency. We have two distinct tangent lines with the same slope, meaning the circle is between the two tangent lines. Each corresponding radius must be perpendicular to its tangent line and have slope equal to the negative reciprocal of the two tangent lines. The two radii are distinct, have the same slope, and coincide at the center of the circle, which means the two radii combined form a diameter of the circle. We know the two endpoints of the diameter, so we can determine its slope: $(13 - 5)/(8 - 2) = 8/6 = 4/3$. The two tangents lines are perpendicular to this diameter and have slope equal to the negative reciprocal of the diameter's slope, thus $\mathbf{-3/4}$.



27. Let the four values be $10 \le a \le b \le c$. The mean is $\frac{10+a+b+c}{4}$. The median is $\frac{a+b}{2}$. The range is $c - 10$. The mean being equal to the median implies $\frac{10+c}{4} = \frac{a+b}{4}$, so $10 + c = a + b$. The median being equal to the range implies $a + b = 2(c - 10) = 2c - 20$. Therefore, $10 + c = 2c - 20$, so $c = \mathbf{30}$.

28. Most MATHCOUNTS students are familiar with the intersecting chords theorem. The basis of the proof for that theorem is that triangles ADE and CBE are similar. The scale factor of similarity in this case is that segment BC = 5 and corresponding segment AD = 3, so the linear dimensions of CBE are 5/3 times the corresponding linear dimensions of ADE, and the area of CBE is $(5/3)^2 = 25/9$ times the area of ADE. We can find the area of ADE by Heron's formula since we know the lengths of all three sides: 3, 3 and 4. The semi-perimeter is $(3 + 3 + 4)/2 = 5$, so the area of ADE is given by: $\sqrt{5(5-3)(5-3)(5-4)} = \sqrt{20} = 2\sqrt{5}$. The area CBE is 25/9 times this, or $\mathbf{\frac{50\sqrt{5}}{9}}$ units².



29. $\frac{1}{6} = \left|\frac{x-2018}{x-2019}\right| = \left|1 + \frac{1}{x-2019}\right|$.

The absolute value introduces two cases:

Case 1: $1 + \frac{1}{x-2019} = \frac{1}{6}$, so $\frac{1}{x-2019} = \frac{1}{6} - 1 = -\frac{5}{6}$ and $x - 2019 = -\frac{6}{5}$.

Therefore, $x = 2019 - \frac{6}{5}$. (I am leaving it in this form for reasons to be seen shortly.)

Case 2: $1 + \frac{1}{x-2019} = -\frac{1}{6}$, so $\frac{1}{x-2019} = -\frac{1}{6} - 1 = -\frac{7}{6}$ and $x - 2019 = -\frac{6}{7}$.

Therefore, $x = 2019 - \frac{6}{7}$.

Combining cases: We want $\left|\left(2019 - \frac{6}{5}\right) - \left(2019 - \frac{6}{7}\right)\right| = \left|\frac{6}{7} - \frac{6}{5}\right|$ since the 2019's cancel each other.

$\left|\frac{6}{7} - \frac{6}{5}\right| = \left|\frac{6\times5-6\times7}{7\times5}\right| = \left|-\frac{12}{35}\right| = \mathbf{\frac{12}{35}}$.

30. Each die can independently show any integer from 1 to 6, so the sum S can range from 6 (all 1s) to 36 (all 6s).

For $S = 36$, $S(42 - S) = 36 \times 6 = 216$ uses all 6s with probability $\frac{6!}{6!}\left(\frac{1}{6}\right)^6 = 1\left(\frac{1}{6}\right)^6$.

For $S = 35$, $S(42 - S) = 35 \times 7 = 245$ uses five 6s and one 5 with probability $\frac{6!}{5!1!}\left(\frac{1}{6}\right)^6 = 6\left(\frac{1}{6}\right)^6$.

For $S = 34$, $S(42 - S) = 34 \times 8 = 272$ uses *either* five 6s and one 4 with probability $6\left(\frac{1}{6}\right)^6$ or four 6s and two 5s with probability $\frac{6!}{4!2!}\left(\frac{1}{6}\right)^6 = 15\left(\frac{1}{6}\right)^6$.

For $S = 33$, $S(42 - S) = 33 \times 9 = 297$ does not qualify.

Remember, commutativity of multiplication applies and the reverse cases apply, such as 6×36 for $S = 6$. The same probability applies for $S = 6, 7$ and 8 as for $S = 36, 35$ and 34, respectively, so we need to double the sum of the first three case sets. Therefore, the total probability is

$2(1 + 6 + 6 + 15)\left(\frac{1}{6}\right)^6 = 56\left(\frac{1}{6}\right)^6 = 7 \times 2^3\left(\frac{1}{6}\right)^3\left(\frac{1}{6}\right)^3 = 7\left(\frac{1}{3}\right)^3\left(\frac{1}{6}\right)^3 = \frac{7}{18^3}$

$= \mathbf{\frac{7}{5832}}$.

2018 Chapter Competition Target Round

1. First we must remember that 1 hour = 60 minutes.
   The starting time is at 11:43 a.m., and 17 minutes after that, we reach 60 minutes after 11:00 a.m., which is 12:00 noon.
   From noon to 2:34 p.m. is 2 hours + 34 minutes = 2 × 60 minutes + 34 minutes = 120 minutes + 34 minutes = 154 minutes.
   The total time is the sum of the morning time, 17 minutes, and the afternoon time, 154 minutes, which is **171** minutes.

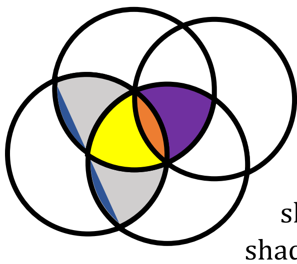
2. We can factor the quadratic $2x^2 + 5x - 12$ into $(x + 4)(2x - 3)$. Substituting and solving the quadratic equation yields $(x + 4)(2x - 3) = 0$, so $x + 4 = 0$ and $x = -4$, or $2x - 3 = 0$ and $x = \frac{3}{2}$. Now, let $m = -4$ and $n = \frac{3}{2}$. Substituting into the expression $(m - 1)(n - 1)$, we get $(-4 - 1)\left(\frac{3}{2} - 1\right) = (-5)\left(\frac{1}{2}\right) = -\mathbf{\frac{5}{2}}$.

3. Determine x given: $x > y$; $x + y = 35$; $x = 2y - 4$.
   $y = 35 - x$ and $x = 2y - 4$ imply that $x = 2(35 - x) - 4 = 70 - 2x - 4 = 66 - 2x$, so $3x = 66$. Therefore, $x = \mathbf{22}$.

4. There are 4!/(2! × 2!) = 6 distinct arrangements of the 4 values into the 2 pairs. The division by 2! twice in this count is based on $a + b = b + a$, yielding a duplicate outcome, and likewise for $c + d = d + c$. These 6 arrangements are:
   ((1+2), (3+4)) = (3, 7);
   ((1+3), (2+4)) = (4, 6);
   ((1+4), (2+3)) = (5, 5);
   ((2+3), (1+4)) = (5, 5);
   ((2+4), (1+3)) = (6, 4);
   ((3+4), (1+2)) = (7, 3).
   We see that (5, 5) is duplicated, so there are only 5 distinct cases at this level. Thus, we are looking at evaluating 3^7, 4^6, 5^5, 6^4, and 7^3. There is a theorem relating x^y and y^x for e < x < y, where e is a special irrational number (like π) with value 2.718…: $y^x < x^y$. We want the maximum power out of these 5 choices. We know from the theorem that $3^7 > 7^3$ and $4^6 > 6^4$, so we are down to 3 powers to evaluate: $3^7 = 2187$, $4^6 = 4096$, and $5^5 = 3125$, the greatest of which is **4096**.

5. Let a and b be the number of laps that Aiden and Bryce, respectively, make when they meet again at the starting line. Because the requisite meeting place is at the starting line, that involves a whole number of laps for each, thus making both a and b to be integers. The times involved are $44a$ seconds and $40b$ seconds, which must be equal to be a meeting.
   Now, $44a$ seconds = $40b$ seconds, and manipulating algebraically yields $b/a = 44/40 = 11/10$, which cannot be reduced further. Therefore, Bryce takes 11 laps while Aiden takes 10 laps, corresponding to 10 × 44 seconds = 11 × 40 seconds = **440** seconds.

6. The movie lasts 2 hours + 18 minutes = 2 × 60 minutes + 18 minutes = 120 minutes + 18 minutes = 138 minutes. An ad is shown at the very beginning and then after each 10 minutes of movie, thus at 0, and after 10 minutes, 20 minutes, …, 130 minutes of movie, which is a total of 14 ads. Each ad takes 30 seconds, which is (1/2) minute, so 14 of them last 14 × (1/2) minutes = 7 minutes. Therefore, the total watching time is 138 minutes + 7 minutes = 145 minutes, of which 7 minutes is advertising, making the fraction of time spent watching ads equal to 7 minutes/(145 minutes) = (7/145) × 100% ≈ **4.8**%.

7. 

The original figure involved finding the area of what is shaded here with blue, gray, and purple. Because all of the circles are congruent, there are various corresponding regions. The purple region is congruent to the yellow region. The two blue regions joined flat side to flat side is congruent to the orange region. If we move the purple shading to the yellow shading and the two blue shadings to the orange shading, so that we assess only the gray, yellow, and orange shading, we see that we have a semicircle made up of half of one of the four congruent circles.
Therefore, the area of the shaded region is $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(2\text{ cm})^2 = \mathbf{2\pi}\text{ cm}^2$.

8. In order to have $3 - \sqrt{\frac{4-x}{2}}$ to be meaningful at all in the context of real numbers, the radicand, $(4 - x)/2$, must not be negative, so $x \leq 4$. We must also keep the radical at most 3, so the radicand must be at most $3^2 = 9$; therefore, $4 - x \leq 18$, that is, $x \geq -14$.
Therefore, we wish to add the integers from −14 through +4. This is a simple arithmetic series to add using the standard formula; one might also note that 1 through 4 cancel out −1 through −4, so add −14 through −5; as yet another option, adding 1 through *n*, or the negative thereof, is the simplest of the arithmetic series formulas and separately do −14 through −1 and 1 through 4. I prefer the last option:
−14 through −1 is the negative of 1 through 14: −(14 × 15/2) = −7 × 15 = −105, and 1 through 4 is easily found to be 10, so −105 + 10 = **−95**.

2018 Chapter Competition Team Round

1. Let Q represent a quarter, D a dime, N a nickel, and P a penny. The absence of a letter in an expression indicate no corresponding coin is present; for coins that are present, a subscript applied to a letter for a coin indicates how many of that coin type is there. Let's start with as big a coin as possible and gradually break it down to smaller-valued coins:
   Q_1P_2: 3 coins, which is odd. ✓
   $D_2N_1P_2$: 5 coins, which is odd. ✓
   D_2P_7: 9 coins, which is odd. ✓
   Replacing one D with any of two N, or one N and five P, or ten P in either of the preceding two cases adds an odd number of coins, making the odd count go to even, so none of those work. Those are all the cases that have a Q or D, so all that is left are N and P cases. Start with replacing Q in the first case with 5N, an increase of 4 coins, so the odd case that worked at the beginning works with this replacement:
   N_5P_2: 7 coins, which is odd. ✓
   Now we successively replace one N at a time with five P, increasing the odd count by 4 each time, resulting in a new odd count:
   N_4P_7: 11coins, which is odd; ✓
   N_3P_{12}: 15 coins, which is odd; ✓
   N_2P_{17}: 19 coins, which is odd; ✓
   N_1P_{22}: 23 coins, which is odd; ✓
   P_{27}: 27 coins, which is odd. ✓
   Counting the check marks, there are **9** ways.

2. The sum of the measures of all 6 interior angles of a hexagon is $(6-2)180$ degrees $=$ 720 degrees. Therefore, 720 degrees $= x + 10$ degrees $+ 2x + 80$ degrees $+ 3x -$ 60 degrees $+ 4x + 40$ degrees $+ 5x - 10$ degrees $+ 6x - 33$ degrees $= 21x + 27$ degrees, and 693 degrees $= 21x$, so $x = 33$ degrees. That results in the following angles:
   33 degrees + 10 degrees = 43 degrees;
   2×33 degrees + 80 degrees = 146 degrees;
   3×33 degrees − 60 degrees = 39 degrees;
   4×33 degrees + 40 degrees = 172 degrees;
   5×33 degrees − 10 degrees = 155 degrees;
   6×33 degrees − 33 degrees = 165 degrees.
   The largest of the values is **172** degrees.

3. To get the greatest possible sum, we need to put the largest value, 21, in the circle used every time, the center circle. On any branch the sum of the two outer circles must match the sum of the least and greatest remaining numbers after removing the 21, thus 11 + 20 = 31. The maximum possible sum per the specified criteria is, therefore, 21 + 31 = **52**.

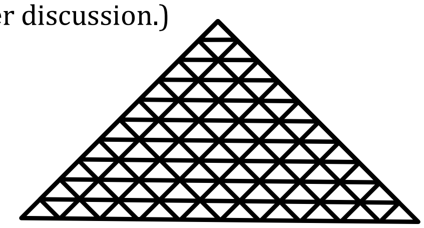
4. $(ax+by)(cx+dy) = acx^2 + (ad+bc)xy + bdy^2 = 7x^2 + 9kxy + 5y^2$.
Therefore, $ac = 7$, so $a = 1$ and $c = 7$, or $a = 7$ and $c = 1$;
similarly, $bd = 5$, so $b = 1$ and $d = 5$, or $b = 5$ and $d = 1$.
We must take each of the two possible pairings of a and c and combine with each of the two pairings of b and d to see how we can get $ad + bc$ to be a multiple of 9. The options are:
$1 \times 5 + 1 \times 7 = 12$—not a multiple of 9;
$7 \times 5 + 1 \times 1 = 36$—a multiple of 9;
$1 \times 1 + 5 \times 7 = 36$—a multiple of 9;
$7 \times 1 + 5 \times 1 = 12$—not a multiple of 9.
There are two cases that work, both with value 36, so $9k = 36$ and $k = \mathbf{4}$.

5. Let c be the cost of the child ticket; then the cost of a senior ticket is double that, or $2c$, and regular ticket is doubled yet again, or $4c$. There are 6 child tickets, 8 senior tickets, and 55 regular tickets. The total price is $\$544.50 = 6c + 8 \times 2c + 55 \times 4c = 242c$; therefore, $c = \$544.50/242 = \$\mathbf{2.25}$.

6. For a fraction to have value 0, the numerator must be 0; thus, the numerator must be $x - 5$ (the numerator being x minus something and it must be 0 for $x = 5$, so the something must be 5, regardless whether it is called $3a$ or anything else—no need to calculate a or $3a$.) Likewise, being undefined at $x = 3$ means the denominator is 0 at $x = 3$ so the denominator is $x - 3$.
Therefore, $f(x) = \frac{x-5}{x-3} = 1 - \frac{2}{x-3}$, and $f\left(\frac{1}{3}\right) = 1 - \frac{2}{\frac{1}{3}-3} = 1 - 2\left(-\frac{3}{8}\right) = 1 + \frac{3}{4} = \mathbf{\frac{7}{4}}$.

7. We know that 2 and 3 are keys, and there is a third, as yet unknown key. Because we want the product of all three elements in the set that contains neither 2 nor 3, that unknown key must be determined.
5 and 7 go in the same set, but neither is a key, so those two values go with one of the three keys, and that key is the only thing that is unknown for that set.
Because 13, 17, and 29 must be in different sets, one of them must be the key in the set with 5 and 7.
Now, 13 must be with 11 or 19, which cannot be the case if it is with 5 and 7, so we are down to 17 and 29 for consideration.
7 cannot be in a set with 17 as a key, and we are dealing with the set with 5 and 7, so 17 cannot be the key, so the only value left for the key is 29.
Our set appears to be {29, 5, 7}. A quick check of the other rules yields no contradiction. The desired answer is the product 29 × 5 × 7 = **1015**.

8. (Note: I have re-oriented the figure from the problem for easier discussion.)



We need to treat the triangles oriented △ differently from how we treat triangles oriented ▽ (sorry, but you will need to pretend these are right triangles)—we will see why shortly. There is 1 unit △ in the top row, 2 in the second row, 3 in the third row, ..., and 10 in the tenth (bottom) row. Thus, there are $1 + 2 + 3 + \cdots + 10 = T_{10}$ unit △ in total, where T_k is the kth triangular number, $C_{k+1,2} = \frac{(k+1)k}{2}$. The top two rows together comprise a single "2×2" triangle oriented as △. Go one row down (second and third rows) and there are 2 more. Each time we go down a row, there is 1 more than the previous row, and we can go down to the ninth-tenth row pair, which has 9. Therefore, there are $1 + 2 + 3 + \cdots + 9 = T_9$ such "2×2" △. We can keep making the component triangle bigger by 1 more row at a time, and this pattern continues: count of T_8 for "3×3" △, T_7 for "4×4", ..., until finally $T_1 = 1$ for "10×10" when we consider the whole given triangle as a component triangle. This yields a total of count of △ of any size of $T_1 + T_2 + T_3 + \cdots T_{10} = C_{12,3} = 12 \times 11 \times 10/6 = 220$. (This is a basic property of Pascal's triangle—the sum of the top n 1's $[C_{k,0}]$ in the leftmost diagonal is the nth positive integer n $[C_{n,1}]$, the sum of the first n positive integers $[C_{k,1}]$ in the next diagonal is the nth triangular number $(n+1)n/2 = C_{n+1,2}$, the sum of the first n triangular numbers $[C_{k+1,2}]$ in the next diagonal is the nth tetrahedral number $(n+2)(n+1)n/3! = C_{n+2,3}$, and so on.)

Now we have to deal with the triangles oriented ▽. Such a triangle does not occur until the second row, and there is 1; each successive row (of which there are 8) adds 1 more unit ▽, so we have $1 + 2 + 3 + \cdots + 9 = T_9$ unit ▽. The "2×2" ▽ start in row 3 but the bottommost one starts in row 9 (occupying rows 9 and 10), so there only 7 rows with such component triangles yielding a total of T_7 of them—note that we have skipped T_8. This skipping pattern continues: T_5 "3×3", T_3 "4×4", and T_1 "5×5". There is no farther to go—the reason being that the one "5×5" has its top in row 6 and its bottom in row 10 (the last row), so there is no room at the bottom to grow a larger ▽. Thus, the total number of ▽ is $T_1 + T_3 + T_5 + T_7 + T_9$. There is a simple, but not so trivial to remember, formula to evaluate such an expression, plus a different formula must be used if the total number of rows is odd. Instead let's just crunch through the numbers: $\frac{2\times1}{2} + \frac{4\times3}{2} + \frac{6\times5}{2} + \frac{8\times7}{2} + \frac{10\times9}{2} = 1 + 6 + 15 + 28 + 45 = 95$.

Combining the two orientations of triangles, we have a total of $220 + 95 = \mathbf{315}$.
There is a straightforward overall formula that handles the general case for the total rather simply, even though the ▽ portion is messy: $\left\lfloor\frac{n(n+2)(2n+1)}{8}\right\rfloor$. The outer "brackets" are *not* ordinary rectangular brackets (notice the inward horizontal extensions at the bottom but not the top) but constitute the symbol for the floor operator (also known as the greatest integer function). Note that when n is even the numerator is divisible by 8, so no issue; however, when n is odd the numerator is odd and not divisible by 8. However, if you round the quotient *down* (in the negative direction) to an integer, the result always works—whether odd or even. Without this rounding trick, the formula for the odd n case would look quite different.

9. To make one pound of equal mix, there needs to be one-half pound gummy bears and one-half pound of jelly beans. Gummy bears cost \$20 for 8 pounds, so \$20/(8 lb) = \$2.50/lb, and one-

half pound would cost \$1.25. Jelly beans cost \$14 for four pounds, so \$14/(4 lb) = \$3.50/lb, and one-half pound would cost \$1.75. The total cost to make one pound of the mix is \$1.25 + \$1.75 = \$3.00, which is to be 2/5 (that is, 40 % = 40/100) of the selling price, so the selling price needs to be \$3.00 × 5/2 = \$**7.50**.

10. There are three cases:
Case 1. Both Ds and both Is are used, with 1 of the other 4 letters: With each of two letters used twice and one used once, that yields $\frac{5!}{2!2!1!} = 30$ arrangements. Now for the letter on its own, there are 4 choices (A, T, O, N). Combining those two features yields 30 × 4 = 120 options.
Case 2. Both Ds are used but not both Is, or both Is are used but not both Ds: With one letter used twice and each of three letters used once, that yields $\frac{5!}{2!1!1!1!} = 60$ arrangements. It could be either the D or the I that is doubled, providing a factor of 2, and we can choose any 3 of the 5 distinct remaining letters, for another factor of $\frac{5!}{3!2!} = 10$. Combining these three factors yields 60 × 2 × 10 = 1200 options.
Case 3. There is at most one D and at most one I used: There are 6 distinct letters of which any 5 are to be chosen, yielding $\frac{6!}{(6-5)!} = 720$ options.
Total: These three cases yield a total count of 120 + 1200 + 720 = **2040** permutations.