SOLUTIONS

The solutions provided here are only *possible* solutions. It is very likely that you or your students will come up with additional—and perhaps more elegant—solutions. Happy solving!

Warm-Up 1

1. Subtract the year the capsule was sealed, 1940, from the year it will be opened, 2017. This difference is the amount of time in between, or 2017 − 1940 = **77** years.

2. This can be done using the proportion 5/2 = 8/x, x being the number of large paper clips equivalent in length to 8 small paper clips. To find x, multiply each side of the equation by x and then by 2/5. The result is x = 16/5 large clips, which as a mixed number is $\mathbf{3\frac{1}{5}}$ large paper clips.

3. One path goes directly from cell 1 to cell 7. To find all the other paths, we will systematically list five clockwise paths and then five counterclockwise paths that end in cell 7. The **11** paths are 1-7, 1-2-7, 1-3-2-7, 1-4-3-2-7, 1-5-4-3-2-7, 1-6-5-4-3-2-7, 1-6-7, 1-5-6-7, 1-4-5-6-7, 1-3-4-5-6-7, 1-2-3-4-5-6-7.

4. Dividing by 2/3 is the same as multiplying by 3/2, so 4 ÷ (2/3) − 5 = 4 × (3/2) − 5 = 6 − 5 = **1**.

5. The absolute difference, or the absolute value of the difference, can be thought of as the positive difference. To find the positive difference between the two values, subtract the smaller fraction from the larger fraction as follows: (1/2) − (1/3) = (3/6) − (2/6) = **1/6**.

6. If the perimeter of rectangle ABCD is 34 cm, then the semiperimeter is half that measure, or 17 cm, and side AD must measure 17 − 5 = 12 cm. To find the length of diagonal BD, we use the Pythagorean Theorem as follows: $5^2 + 12^2 = x^2 \rightarrow 25 + 144 = x^2 \rightarrow 169 = x^2 \rightarrow x = 13$. (5-12-13 is one of several Pythagorean Triples memorized by competitive Mathletes.) The perimeter of △ABD is 5 + 12 + 13 = **30** cm.

7. There are 8 × 8 = 64 ways the two rolls of the eight-sided die can occur. This will be the denominator of our probability fraction. If the first roll of the die is a 1, then the second roll can be any of the 8 numbers for it to be greater than or equal to the first roll. If the first roll is a 2, then the second roll can be any of 7 possibilities, excluding 1. With a 3, we can have 6 possibilities, etc. The total number of acceptable rolls for the second die is, thus, 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36. The probability is 36/64 = **9/16**.

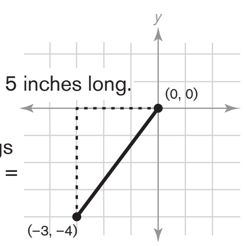
8. The product $1.2 \times 10^3 \times 1.4 \times 10^2$ can be rewritten as $1.2 \times 1.4 \times 10^3 \times 10^2$, which is $1.68 \times 10^{3+2} = 1.68 \times 10^5 \approx \mathbf{1.7 \times 10^5}$.

9. The square of a positive integer is equal to the square of its opposite. For example $5^2 = (-5)^2$. So all the integers n that satisfy the requirement come in pairs of opposites. Therefore, their sum is **0**.

10. If Minnie took 7 lessons, and lessons cost \$20 each, she paid 7 × 20 = \$140, plus the \$30 registration fee, for a total of 140 + 30 = **\$170**.

Warm-Up 2

11. A string that is 7 feet long is 7 × 12 = 84 inches long, which is 16 × 5 + 4. So, Sue can cut **16** pieces of string that are 5 inches long.



12. As the figure shows, the segment from the point (−3, −4) to the origin (0, 0) is the hypotenuse of a right triangle with legs of length 3 and 4 units. This is the well-known 3-4-5 right triangle that satisfies the Pythagorean Theorem as follows: $3^2 + 4^2 = 9 + 16 = 25 = 5^2$. The distance from (−3, −4) to the origin is **5** units.

13. The hexagonal tables that are on the ends seat 5 people each. The other 50 − 2 × 5 = 40 people are seated at hexagonal tables that are in the middle of the row and seat only 4 people each. Thus, 40 ÷ 4 = 10 more tables are needed in between the two end tables. That's a total of 2 + 10 = **12** tables.

14. The three notebooks cost 3 × 1.57 = \$4.71. The change from \$5.00 would be 29¢. The fewest coins Gloria could get in change would be a quarter and four pennies, which is **5** coins.

15. The number 1,000,000,000 is 10^9 as a power of 10. When we multiply numbers in scientific notation, we can add the exponents that have the same base. The mass of 1,000,000,000 fluorine atoms is $10^9 \times 3.16 \times 10^{-23} = 3.16 \times 10^{9-23} = \mathbf{3.16 \times 10^{-14}}$ g.

16. Each of the 3 possible appetizers can be combined with each of the 4 entrées for $3 \times 4 = 12$ meals so far. The two side dishes can be selected in $5 \times 4 \div 2 = 10$ ways. We divide by 2 because the $5 \times 4 = 20$ ways would include duplicates of every pair of side dishes chosen the other way around. We now have $12 \times 10 = 120$ meals with an appetizer, an entrée and two side dishes. Each of these 120 meals can be combined with any of the 6 desserts for $120 \times 6 =$ **720** meals.

17. If 15 machines make 500 raviolis in 15 minutes, then $15 \times 5 = 75$ machines would make $5 \times 500 = 2500$ raviolis in the same amount of time, 15 minutes. To get from 2500 raviolis to 6000 raviolis, those 75 machines would need to run for 6000/2500 or 12/5 times as long and $12/5 \times 15 =$ **36** minutes.

18. Let's list all $4 \times 3 = 12$ possible two-digit numbers that can be made from the four digits and count those that are prime: 12, <u>13</u>, <u>17</u>, 21, <u>23</u>, 27, <u>31</u>, 32, <u>37</u>, <u>71</u>, 72, <u>73</u>. Seven of these numbers are prime, so the probability is **7/12**.

19. The two letters that are 4 units from the letter M are I and Q. These are the 9th and 17th letters in the alphabet, so they would be placed at 8 and 16 on the number line. Not surprisingly, the average of these two numbers is exactly the number of M, which is **12**.

20. The location of point B is $(-2, -2)$, but we don't really need to know this. The more important information is the lengths of the legs of the right triangle that can be formed on the grid with points A and B at opposite ends of the hypotenuse, and these lengths were given. Using the Pythagorean Theorem, we can find the length of segment AB as follows: $2^2 + 8^2 = x^2 \rightarrow 4 + 64 = x^2 \rightarrow 68 = x^2 \rightarrow x = \sqrt{68} = \sqrt{(4 \times 17)} = \sqrt{4} \times \sqrt{17} = \mathbf{2\sqrt{17}}$ units.

Warm-Up 3

21. The average of an arithmetic sequence is the average of its first and last terms. In this case, we have $(13 + 31) \div 2 = 44 \div 2 =$ **22**.

22. Suppose Kelly first selects a white sock and then selects a black sock. The third sock she selects must match either the white or the black, forming a pair either way. Thus, she must select **3** socks to be assured of a matching pair.

23. The fraction 27/72 expressed as a common fraction is 3/8. So m must be **8**.

24. The total number of chairs in the room must be $23 \times 27 =$ **621**. Most students will use the standard algorithm for multiplying two-digit integers. But to quickly multiply these two numbers, we can write the product as the difference of two squares: $(25 - 2)(25 + 2) = 25^2 - 2^2 = 625 - 4 =$ **621**.

25. Without equations, we might think about this problem as follows: Since the larger number is 4 times the smaller number, then the sum of the two numbers must be 5 times the smaller number. This same sum is 3 times the smaller number plus 18, so 18 must be 2 times the smaller number and the smaller number is **9**. If we write equations, we have $y = 4x$ and $x + y = 3x + 18$. Replacing the y in the second equation with $4x$, we get $x + 4x = 3x + 18 \rightarrow 5x = 3x + 18 \rightarrow 2x = 18 \rightarrow x =$ **9**. This matches our earlier thinking exactly.

26. The ratio of the hypotenuse of $\triangle$ABC to the hypotenuse of $\triangle$XYZ is 10/18, or 5/9. This is a one-dimensional scale factor. The area of the triangles is two-dimensional, so we have to use this scale factor twice. The ratio we want is $5/9 \times 5/9 =$ **25/81**.

27. There are seven numbers from 1 through 25, inclusive, that are *not* divisible by 2, 3, 4 or 5. They are 1, 7, 11, 13, 17, 19 and 23. The other $25 - 7 = 18$ numbers *are* divisible by one or more of 2, 3, 4 or 5, so the probability is **18/25**.

28. The **8** ordered sequences that Siddarth can use to write 5 as a sum of 1s and 2s are $1 + 1 + 1 + 1 + 1$, $2 + 1 + 1 + 1$, $1 + 2 + 1 + 1$, $1 + 1 + 2 + 1$, $1 + 1 + 1 + 2$, $2 + 2 + 1$, $2 + 1 + 2$ and $1 + 2 + 2$. In general, the number of distinct ordered sequences containing only 1s and 2s and having a sum of n is the nth Fibonacci number, where the first and second terms are 1 and 2, respectively.

29. Kevin's bus traveled 15 miles west and 8 miles north for a total of 23 miles. The direct path would be the hypotenuse of a triangle with legs of length 8 miles and 15 miles. Using the Pythagorean Theorem, we can find the length of the hypotenuse as follows: $8^2 + 15^2 = x^2 \rightarrow 64 + 225 = x^2 \rightarrow 289 = x^2 \rightarrow x = 17$. (This is another Pythagorean Triple that serious Mathletes should memorize.) Kevin's bus would travel 17 miles on the direct route, which is shorter by $23 - 17 =$ **6** miles.

30. Ellen would get the following sequence of calculations: $123{,}456 - 6 = 123{,}450 \rightarrow 123{,}450 \div 10 = 12{,}345 \rightarrow 12{,}345 - 5 = 12{,}340 \rightarrow 12{,}340 \div 10 = 1234 \rightarrow 1234 - 4 = 1230 \rightarrow 1230 \div 10 = 123 \rightarrow 123 - 3 = 120 \rightarrow 120 \div 10 = 12 \rightarrow 12 - 2 = 10 \rightarrow 10 \div 10 =$ **1.**

Workout 1

31. The exchange rate in the other direction is the reciprocal of 0.77, or $1 \div 0.77$, which is 1.2987012.... So US\$1.00 is about NZ\$**1.30**.

32. We are looking for a multiple of 7 that is one more than a multiple of 2, 3, 4, 5, 6 and 8. The LCM of 2, 3, 4, 5, 6 and 8 is $3 \times 5 \times 8 = 120$, so 121 will leave a remainder of 1 if we divide by 2, 3, 4, 5, 6 or 8. Unfortunately 121 is not a multiple of 7; it's 2 more than a multiple of 7. Let's try $2 \times 120 + 1 = 241$. This is 3 more than a multiple of 7 or 3 (mod 7). The pattern continues with 361 = 4 (mod 7), 481 = 5 (mod 7) and 601 = 6 (mod 7), but finally 721 = 0 (mod 7). The least number of pennies that could be in the jar is **721** pennies.

33. For the first through fifth years, we have populations of n, then $3n$, then $3n - 3000$, then $(3n - 3000)/2$, then finally $(3n - 3000)/2 + 1300 = 1450$ lemmings. Now, starting at the fifth year and working backwards, we have $(3n - 3000)/2 = 150$, then $3n - 3000 = 300$, then $3n = 3300$, then at the start $n =$ **1100** lemmings.

34. With no restrictions, $4 \times 3 \times 2 \times 1 = 24$ positive integers can be formed from the four given digits. The restriction means that we cannot have a 1 next to a 4. There are 3 ways to place a 4 and a 1 next to each other; as the first two digits, the middle two digits and the last two digits. There are 2 ways to order the 4 and the 1 and 2 ways to order the 3 and the 2, for $3 \times 2 \times 2 = 12$ integers. This eliminates 12 of the 24 integers and leaves **12** integers.

35. A tire with a diameter of 25 inches has a circumference of 25π inches, so it's covering 25π inches per rotation. Here is a single expression that uses units cancellation to convert 65 miles per hour to the number of tire rotations per second.

$$\frac{1\text{ rotation}}{25\pi\ \cancel{\text{inches}}} \times \frac{12\ \cancel{\text{inches}}}{1\ \cancel{\text{foot}}} \times \frac{5280\ \cancel{\text{feet}}}{1\ \cancel{\text{mile}}} \times \frac{65\ \cancel{\text{miles}}}{1\ \cancel{\text{hour}}} \times \frac{1\ \cancel{\text{hour}}}{60\ \cancel{\text{minutes}}} \times \frac{1\ \cancel{\text{minute}}}{60\text{ seconds}} = \frac{12 \times 5280 \times 65\text{ rotations}}{25\pi \times 60 \times 60\text{ seconds}} \approx \mathbf{14.6}\text{ rotations per second}$$

36. To increase a number by 10%, multiply by 1.1. In week 2, Nish runs $8 \times 1.1 = 8.8$ miles, as stated. In week 3, she runs $8.8 \times 1.1 = 9.68$ miles. In week 4, she runs $9.68 \times 1.1 = 10.648$ miles. In week 5, she runs $10.648 \times 1.1 = 11.7128$ miles. In week 6, she runs $11.7128 \times 1.1 = 12.88408$ miles. In week 7, she runs $12.88408 \times 1.1 = 14.172488$ miles. The first week of training in which Nish exceeds 13.1 miles is week **7**.

37. We are told that the 8 from 18 follows the A in the second time through the alphabet. There are 8 more two-digit numbers from 19 to 26, inclusive. This is 16 more digits that must follow 16 more letters after the A. We want the 17th letter of the alphabet, which is **Q**.

38. The area of the original rectangle is $4 \times 5 = 20$ cm^2. The dodecagon is 4 cm^2 less, so it's **16** cm^2.

39. The sum of the first ten positive integers is $10 \times 11 \div 2 = 55$. The nearest perfect square less than 55 is 49, sothe **6** must not have been included in the sum.

40. If we want the green marbles to be 5/12 of the marbles, then the 40 green marbles must be 5 parts, so each part is $40 \div 5 = 8$. The total number of marbles needs to be $8 \times 12 = 96$. There are $20 + 40 = 60$ marbles so far, so $96 - 60 =$ **36** purple marbles must be added.

Workout 2

41. The second cube of cheese weighs $64.8 \times 16 = 1036.8$ ounces. This type of cheese weighs 0.6 ounce per cubic inch. So the cube of cheese weighing 1036.8 ounces would have volume $1036.8 \div 0.6 = 1728$ in^3 and edge length $\sqrt[3]{1728} = 12$ inches = **1** foot.

42. Only the perfect squares have an odd number of divisors. There are $\sqrt{100} = 10$ perfect squares from 1 to 100, inclusive. That means there are $100 - 10 =$ **90** integers with an even number of factors.

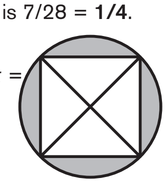
43. The distance from -1.3 to 3 1/8, or 3.125, is $1.3 + 3.125 = 4.425$ units. One-third of this distance is 1.475 and 2/3 is 2.95. If we add 2.95 to -1.3 or subtract 1.475 from 3.125, we get to the same point that is 2/3 of the way between the endpoints: $-1.3 + 2.95 =$ **1.65**.

44. The stick of butter has a volume of $1.5 \times 1.5 \times 3.25 = 7.3125$ in^3. The pat of butter has a volume of $1 \times 1 \times 0.375 = 0.375$ in^3. The number of calories in a pat of butter is $\frac{800\text{ calories}}{1.5 \times 1.5 \times 3.25\ \cancel{\text{in}^3}} \times \frac{1 \times 1 \times 0.375\ \cancel{\text{in}^3}}{1\text{ pat}} \approx$ **41** calories.

45. The two diagonals bisect each other, so triangles AED and BEC must be equilateral, with side lengths of 5 and angles of 60 degrees. The measure of $\angle AEB$ is the supplement of 60, which is $180 - 60 =$ **120** degrees.

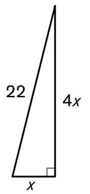
46. The expression 4! + 5! can be rewritten as $4!(1 + 5) = 6 \times 4 \times 3! = 24 \times 3!$. This means that $n! = 24$, so $n =$ **4**.

47. There are 28 dominoes in a full set. There are $7 \times 6 \div 2 = 21$ dominoes with different numbers of dots on the two sides of the line and 7 with the same number of dots on both sides of the line, called doubles. The probability that one of the doubles is selected at random is 7/28 = **1/4**.



48. Since we are looking for a percent, we can assign a radius of 1 unit to our circular pizza. The area of the pizza is then $1^2 \times \pi = \pi$ units^2. The inscribed square can be cut into four isosceles right triangles that have legs of 1 unit. Each triangle has an area of 1/2 units^2 and the four of them can be rearranged to form 2 unit squares. The difference $\pi - 2$ is the area of the four segments that remain when Paige removes the square. As a percent, this is $(\pi - 2)/\pi \times 100\% \approx$ **36**%.

49. If we combine what Jefferson and Monroe exchange, we get a total of 7 ziggles and 7 zoggles exchanged for a total of 49 zaggles. Dividing these totals by 7, we find that Carter can expect to exchange 1 ziggle and 1 zoggle for **7** zaggles.



50. Assuming that the ground is perpendicular to the vertical structure, we can draw a right triangle with legs of x and $4x$ feet and a hypotenuse of 22 feet. We can then use the Pythagorean Theorem and solve for x as follows: $x^2 + (4x)^2 = 22^2 \rightarrow 1x^2 + 16x^2 = 484 \rightarrow 17x^2 = 484 \rightarrow x^2 = 484/17 \rightarrow x = \sqrt{(484/17)} \rightarrow x \approx 5.336$. This is the value of x, which tells us how far the base of the ladder can be from the wall. We want to know how high the ladder can safely reach up the vertical structure, which is $4x$, or about $4 \times 5.336 \approx$ **21.3** feet.

Warm-Up 4

51. From least to greatest, Jamie's first five tests scores were 75, 81, 86, 92, 98. We want to maximize the value of the lowest of the next three test scores so that the median of all eight scores is 88. The median will be the average of the 4th and 5th scores in an ordered list of the eight scores. There is no way to get a median of 88 with two of the five known scores being the 4th and 5th scores. We could let two of the unknown scores be 87 and 89 to get a median of 88. In this case the third unknown score just has to be greater than or equal to 89, and the lowest of these three scores is 87. We could let 86 be the 4th score and make the 5th score 90 to get a median of 88. Then the lowest of the three unknown scores would be 86. If we let the 4th and 5th scores both be 88, we get a median of 88, and the final unknown score can have any value greater than or equal to 88. Further analysis shows that if the lowest of the unknown scores is greater than 88, the median would be 89 or more. Therefore, the greatest possible value of the lowest of the final three scores is **88**.

52. The least and greatest possible two-digit primes containing only 1s, 3s, 7s and 9s as digits are 11 and 97, respectively. The absolute difference between these two values is $97 - 11 =$ **86**.

53. The sum of each pair of numbers is twice the mean of the pair. From the information given, we can write the following system of equations: $a + b = 16$, $b + c = 32$ and $a + c = 28$. Adding these three equations, we get $2a + 2b + 2c = 76$. So $a + b + c$ will be half of 76, or **38**.

54. The difference between the length and the width of the rectangle is $(4x + 9) - (4x - 3) = 4x + 9 - 4x + 3 =$ **12** units.

55. Each of the six faces of the cube can be painted one of two colors, so there are $2^6 = 64$ equally likely possible ways to color the six faces. Some of these ways will look the same under rotation, but only one of them is all blue, so the probability is **1/64**.

56. We first evaluate $n^2 - n$ for $n = 5$ to get $25 - 5 = 20$. Now we evaluate $n^2 - n$ for $n = 20$ to get $400 - 20 =$ **380**.

57. The possible two-digit primes that use only the digits 1 through 8 are 11, 13, 17, 23, 31, 37, 41, 43, 47, 53, 61, 67, 71, 73 and 83. That's 15 out of $8 \times 8 = 64$ possible outcomes when an eight-sided die is rolled twice. So the probability is **15/64**.

58. Substituting the given values of x and y, we get $2 \times 3^2 + 3 \times (-2)^2 - 4 \times 3 + 2 \times (-2) - 17 = 2 \times 9 + 3 \times 4 - 12 - 4 - 17 = 18 + 12 - 12 - 21 = 18 - 21 =$ **−3**.

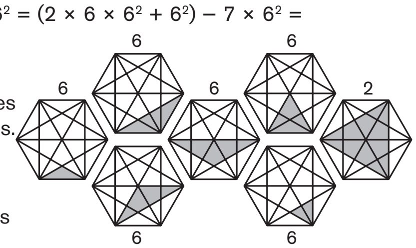
59. The number of possible old-style plates is $25 \times 25 \times 10 \times 10 \times 10 \times 10$. The number of possible new-style plates is $25 \times 25 \times 25 \times 10 \times 10 \times 10$. The ratio is $(25 \times 25 \times 10 \times 10 \times 10 \times 10)/(25 \times 25 \times 25 \times 10 \times 10 \times 10) = 10/25 =$ **2/5**.

60. Since $40\% = 2/5$ and $0.8 = 4/5$, we have $2/5 \times 2/3 \times 24 \div 4/5 = (2 \times 2 \times 24 \times 5)/(5 \times 3 \times 4) =$ **8**.

Warm-Up 5

61. We will factor out a common factor of 6^2 to make the calculations simpler. We have $(2 \times 6^3 + 6^2) - 7 \times 6^2 = (2 \times 6 \times 6^2 + 6^2) - 7 \times 6^2 = 6^2 \times (12 + 1 - 7) = 6^2 \times 6 = 6^3 =$ **216**.

62. There are seven types of triangles in the figure that have at least two sides congruent. As shown, six types occur in six different locations and one type occurs in just two locations. That's a total of $6 \times 6 + 2 =$ **38** triangles.



63. This problem is similar to problem 28, which asked in how many ways Sidarth could write 5 as a sum of 1s and 2s. Since Fido never climbs more than three stairs in one step, instead of just partitioning the stairs into groups of 1 and 2, we can also include groups of 3.
There is <u>1</u> way for Fido to climb the stairs if he steps on each stair (1 + 1 + 1 + 1 + 1).
There are <u>4</u> ways for Fido to climb the stairs if he steps on exactly four stairs (2 + 1 + 1 + 1; 1 + 2 + 1 + 1; 1 + 1 + 2 + 1; 1 + 1 + 1 + 2).
There are <u>6</u> ways for Fido to climb the stairs if he steps on exactly three stairs (2 + 2 + 1; 2 + 1 + 2; 1 + 2 + 2; 3 + 1 + 1; 1 + 3 + 1; 1 + 1 + 3).
These are the 1 + 4 + 6 = **11** ways Fido can climb the five stairs.

64. The x^2 term in the numerator "cancels" the x^2 in the denominator since x^2/x^2 is 1. That leaves $(x^2)^3 = x^2 \cdot x^2 \cdot x^2 = x^6$. So $y =$ **6**.

65. The \$10,000 collected in tax is 8% of some unknown amount, say x dollars, in sales, and we can write the equation $10{,}000 = 0.08x$. Solving, we get $x = 125{,}000$, which means that \$125,000 in sales was subject to the 8% sales tax, and the remaining \$400,000 − \$125,000 = **\$275,000** in sales was not.

66. To ensure that at least one fish travels through each route to the food, let's assign one fish to each of these three routes: above the pipe, through the pipe and below the pipe. Now we need only consider the possibilities for the remaining 97 fish to choose one of the three routes. If none of the 97 fish go above the pipe, then there are 98 ways to divide the 97 fish between the other two routes. If one of the 97 fish goes above the pipe, then there are 97 ways to divide the remaining 96 fish between the other two routes. We can continue counting in this manner until we reach the cases in which all 97 fish go above the pipe and none go through or below it. We get $98 + 97 + 96 + \cdots + 3 + 2 + 1 = (98 \times 99)/2 = 49 \times 99$. At this point, instead of multiplying this out by hand, it would be simpler to calculate $49 \times (100 - 1)$. Doing so, we see that there are $49 \times (100 - 1) = 4900 - 49 =$ **4851** ways for the school of 100 fish to get to the food.

67. Some Mathletes may have already learned that 60 mi/h is 88 ft/s. These students will quickly realize that 44 ft/s must be $(1/2) \times 60 =$ **30** mi/h. Others might convert the speed as follows: $\frac{44\ \cancel{\text{feet}}}{1\ \cancel{\text{second}}} \times \frac{60\ \cancel{\text{seconds}}}{1\ \cancel{\text{minute}}} \times \frac{60\ \cancel{\text{minutes}}}{1\ \text{hour}} \times \frac{1\ \text{mile}}{5280\ \cancel{\text{feet}}} = (44 \times 60 \times 60)/5280 = (44 \times 60)/88 = (22 \times 30)/22 =$ **30** mi/h.

68. The word CHAIRS has six letters with no letters that repeat. So there are $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 =$ **720** ways the letters can be arranged.

69. Let PQ = $2a$ cm. Then PS is $3a$ cm, so PQRS has perimeter $2(2a + 3a) = 10a$ cm and area $(2a)(3a) = 6a^2$ cm^2. We are told that the area of PQRS is 1536 cm^2, so $6a^2 = 1536$. Solving for a, we get $a^2 = 1536/6 \rightarrow a^2 = 256 \rightarrow a = 16$. Then PQRS has perimeter $10a = 10 \times 16 =$ **160** cm.

70. To evaluate a tower of exponents, we start at the top and work our way down. For the first term, $2^{0^{1^2}}$, since $1^2 = 1$ and $0^1 = 0$, we evaluate $2^0 = 1$. For the second term, $2^{2^{1^0}}$, since $1^0 = 1$ and $2^1 = 2$, we evaluate $2^2 = 4$. Therefore, the final result is $1 - 4 =$ **−3**.

Warm-Up 6

71. Since 1 quart = 32 fluid ounces, a quart is 8 times the 4 fluid ounces of lotion that costs \$3, so the cost of a quart will be 8 times \$3, or **\$24**.

72. The number in the bottom right box is the product of the original four numbers. This means that $18 \times a \times 4 \times b = 5184$, so $72ab = 5184$ and $ab = 5184/72 = 72$. We are looking for the least possible value of $a + b$, which occurs when a and b are 8 and 9. We have $8 + 9 =$ **17**.

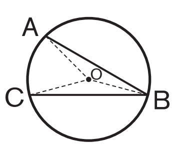
73. As shown, the number 25 is at the center, or the median, of the seventh row. Since the numbers are consecutive, we know that 25 is the average of the seven numbers in that row, so the total of the seven numbers must be $7 \times 25 =$ **175**.

1
2 3
4 5 6
7 8 9 10
11 12 13 14 15
16 17 18 19 20 21
22 23 24 25 26 27 28

74. The terms of an arithmetic sequence must increase or decrease by a constant amount, called the common difference. In this sequence, there must be 5 equal increases from the first term, 12, to the sixth term, 47. Therefore, the common difference must be $(47 - 12) \div 5 = 35 \div 5 = 7$. From the first term, 12, to the fourth term, y, this common difference would be applied 3 times to get $y = 12 + 3(7) = 12 + 21 =$ **33**.

75. Since the width of the rectangle is one-third of its length, then the perimeter must be 1 + 3 + 1 + 3 = 8 widths. Dividing 136 cm by 8, we see that the width is 17 cm, which means the length is 3 × 17 cm = 51 cm. Thus, the area of the rectangle is $17 \times 51 =$ **867** cm^2.

76. This problem is a little tricky because of the three occurrences of the letter A in the word ALASKA. We will consider different cases based on the number of As present and create organized lists. First, let's consider three-letter permutations with three As. There is only $\underline{1}$ such permutation (AAA). Next, let's determine the number of three-letter permutations with two As. There are 3 choices for the third letter, and each three-letter combination can be arranged 3 different ways for a total of $3 \times 3 = \underline{9}$ permutations (AAL, ALA, LAA, AAS, ASA, SAA, AAK, AKA, KAA). If we consider three-letter permutations with one A, there are 3 choices for the other two letters, and there are 3 ways to choose two of these three letters. Each three-letter combination can be arranged in 6 different orders for a total of $3 \times 6 = \underline{18}$ permutations (ALS, ASL, LAS, LSA, SAL, SLA, AKS, ASK, KAS, KSA, SAK, SKA, AKL, ALK, KAL, KLA, LAK, LKA). Finally, we need to count the three-letter permutations with no As. There are 3 choices for the three letters and $3 \times 2 = \underline{6}$ arrangements (KLS, KSL, LKS, LSK, SKL, SLK). That's a total of $1 + 9 + 18 + 6 =$ **34** permutations.



77. If we draw three radii from center O to A, B and C, as shown, we create congruent isosceles triangles AOB and COB. Knowing $m\angle ABC = 30$ degrees, it follows that $m\angle OBC = m\angle OCB = m\angle OAB = m\angle OBA = 15$ degrees. That also means that $m\angle AOB = m\angle COB = 180 - 2(15) = 150$ degrees and $m\angle AOC = 360 - 2(150) = 60$ degrees. Angle AOC is the central angle that intercepts minor $\overset{\frown}{AC}$. Since it is 60/360 = 1/6 of the circle, it follows that the length of $\overset{\frown}{AC}$ is 1/6 of the circle's circumference. The circumference of a circle of diameter 12 inches is $C = d \times \pi = 12\pi$ inches. Therefore, minor $\overset{\frown}{AC}$ has length $(1/6) \times 12\pi =$ **2π** inches. This problem can be solved much faster if you know that the degree measure of inscribed angle ABC is half the measure of its intercepted $\overset{\frown}{AC}$ (see "Angles and Arcs Stretch" on p. 36). So if $m\angle ABC = 30$ degrees, then the degree measure of $\overset{\frown}{AC}$ is $2 \times 30 = 60$ degrees. Since the degree measure of $\overset{\frown}{AC}$ is 60/360 = 1/6 of the circle, this again leads us to conclude that the length of $\overset{\frown}{AC}$ is 1/6 of the circle's circumference, which we calculated to be **2π** inches.

78. For those who do not recognize that A must be **7** since $0.0049 = (0.07)^2$, one way to solve this problem is by converting 0.0049 to a fraction. We have $\sqrt{(0.0049)} = \sqrt{(49/10{,}000)} = \sqrt{(49)}/\sqrt{(10{,}000)} = 7/100 = 0.07$. Therefore, A is **7**.

79. Let's call the two numbers p and q. From the information given, we can write two equations: $|p - q| = 6$ and $|p^2 - q^2| = 24$, which means that $p - q = 6$ or $p - q = -6$ and $p^2 - q^2 = 24$ or $p^2 - q^2 = -24$. Factoring $p^2 - q^2$, we get $(p - q)(p + q) = 24$ or $(p - q)(p + q) = -24$. We can substitute 6 for $p - q$ in these two equations to get $6(p + q) = 24$ and $6(p + q) = -24$, which means that $p + q = 4$ or $p + q = -4$. (Note: Substituting -6 for $p - q$ in the two equations yields the same result.) Now we know that the difference of the two numbers is 6 or -6 and their sum is 4 or -4. Now we can solve the four systems of equations, as shown, to see that the product is $5 \times (-1) = 1 \times (-5) =$ **-5**.

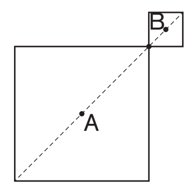
$p+q=4$	$p+q=4$	$p+q=-4$	$p+q=-4$
$\underline{p-q=6}$	$\underline{p-q=-6}$	$\underline{p-q=-6}$	$\underline{p-q=6}$
$2p=10$	$2p=-2$	$2p=-10$	$2p=2$
$p=5 \quad q=-1$	$p=-1 \quad q=5$	$p=-5 \quad q=1$	$p=1 \quad q=-5$

80. We start inside the parentheses and evaluate as follows: 4 ☯ $3 = 4^2 - 2 \times 3^2 = 16 - 18 = -2$. Now we evaluate 5 ☯ $-2 = 5^2 - 2 \times (-2)^2 = 25 - 8 =$ **17**.

Workout 3

81. The average of eleven integers with a sum of 11 must be $11/11 = 1$. Since the integers are consecutive, 1 is also the median. In this case the median is the sixth integer in the ordered list. Therefore, the least of the eleven integers must be $1 - 5 =$ **-4**. A student who doesn't know these properties might approach solving the problem differently. Since the sum of eleven *positive* integers would be greater than 11, we deduce that the consecutive integers in question must include negative values, and when summing these integers, opposites will add to zero, thus leaving only a few consecutive positive integers to add to 11. We can start by considering the eleven integers from -5 to 5, but these integers have a sum of 0. If, however, we consider the integers from -4 to 6, we see that the consecutive integers from -4 to 4 sum to 0, and the final two integers, 5 and 6, sum to 11. Once again we conclude that the least integer is **-4**.

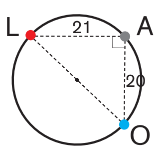
82. The mean of x and y is 12. So the sum of x and y must be $2 \times 12 = 24$, and $x = 24 - y$. The mean of y and 12 is $z/2$. So $(y + 12)/2 = z/2$, and $z = y + 12$. Using these two expressions for x and z, we can write their mean as $(x + z)/2 = (24 - y + y + 12)/2 = 36/2 =$ **18**.



83. Square A has area 225 cm^2, so it must have side length $\sqrt{225} = 15$ cm and diagonal length $15\sqrt{2}$ cm. Square B has area 16 cm^2, so it must have side length $\sqrt{16} = 4$ cm and diagonal length $4\sqrt{2}$ cm. These squares, with centers labeled A and B, are positioned so that A, B and the common vertex are collinear, as shown. This arrangement gives us the greatest distance between A and B, half the sum of the diagonal lengths of the two squares. That distance is $\frac{15\sqrt{2} + 4\sqrt{2}}{2} = \mathbf{\frac{19\sqrt{2}}{2}}$ cm.

84. If we continue the sequence, we get a repeating pattern: 3, 5, 2, –3, –5, –2, 3, 5, 2, –3, –5, –2… We note that the sum of the six repeating terms is 0. So for any integer n that is a multiple of 6, the sum of the first n terms of this sequence will be 0. The greatest multiple of 6 that is less than 200 is 198. So the sum of the first 198 terms will be 0, making the sum of the first 200 terms $3 + 5 =$ **8**.

85. Using divisibility rules, we can quickly determine that 2017 is not divisible by 2, 3 or 5. We might then try dividing by 7, 11, 13, etc., and begin to suspect that 2017 is a prime number. To be sure, we would have to try dividing by all the prime numbers less than or equal to the square root of 2017. Since $45 \times 45 = 2025$, we can be sure that 2017 is prime if it has none of the following primes as factors: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41 and 43. Since none of them is a factor, we conclude that 2017 is prime, and the sum of its factors is $1 + 2017 =$ **2018**.



86. Because of the way that Oberon, Lance and Arthur are seated, a right triangle is formed with legs of length 20 feet and 21 feet. The hypotenuse is a diameter of the circle. You may recognize these two values as being part of the 20-21-29 Pythagorean Triple. Those who do not can use the Pythagorean Theorem to calculate the diameter as $\sqrt{(20^2 + 21^2)} = \sqrt{(400 + 441)} = \sqrt{841} =$ **29** feet.

87. Comparing the two transactions, we see that the extra $12 - 8 = 4$ regular binders in the second transaction have to account for the $46.00 - 32.60 = \$13.40$ difference in cost. Therefore, a regular binder must cost $13.40 \div 4 = \$3.35$. Using this and the information from the first transaction, we calculate the cost of the celebrity binder to be $32.60 - 8 \times 3.35 = 32.60 - 26.80 = 5.80$. The difference between the cost of a celebrity binder and that of a regular binder is $5.80 - 3.35 =$ **\$2.45**.

88. Let p represent the price of a widget. Since Mr. Jones sold n widgets, he earned $0.03 \times p \times n$, and Mr. Smith earned $0.05 \times p \times (n - 500)$. These two commission amounts are equal, so we can write the equation $0.03pn = 0.05p(n - 500) \rightarrow 0.03n = 0.05n - 25$. Solving for n, we get $0.03n = 0.05n - 25 \rightarrow 0.02n = 25 \rightarrow n = 1250$. So Mr. Jones sold 1250 widgets, and Mr. Smith sold $1250 - 500 =$ **750** widgets.

89. The y values in the table appear to be the cubes of the corresponding x values. This would suggest that $a = 1$ and $b = 3$, which results in the sum $a + b = 1 + 3 =$ **4**. Alternatively, from the table of values we have the following equations: $27 = a \times 3^b$ and $8 = a \times 2^b$. Dividing these two equations yields $27/8 = 3^b/2^b$, so $b = 3$, $a = 1$ and $a + b = 1 + 3 =$ **4**.

90. We need to consider the top and bottom surfaces, which are both rings, and the inner and outer lateral surfaces. The top and bottom surfaces each have area $(2.5^2 - 1.5^2)\pi = (6.25 - 2.25)\pi = 4\pi$ in^2. The inner lateral surface has area $2 \times 1.5 \times \pi \times 1 = 3\pi$ in^2. The outer lateral surface has area $2 \times 2.5 \times \pi \times 1 = 5\pi$ in^2. The total surface area Donny calculated for each roll of tape was $4\pi + 4\pi + 3\pi + 5\pi = 16\pi$ in^2. That means the estimated surface area for a dozen donuts is $16\pi \times 12 =$ **192π** in^2.

Workout 4

91. For each of the 3 games Amanda could watch at noon on Saturday, there are 4 games she could watch at 8 p.m. and then 5 games the next day. So there are 3 × 4 × 5 = **60** combinations of games she can watch.

92. When the values of a, b and c are substituted into the expression, we get $\dfrac{\frac{12\times4\times5}{1/2}-(6\times4^2-4)}{0.5}=\dfrac{\frac{240}{1/2}-(96-4)}{0.5}=\dfrac{480-92}{0.5}=\dfrac{388}{0.5}=\mathbf{776}$.

93. The two families are approaching each other at 45 + 53 = 98 mi/h. Since time = distance ÷ rate and the two families start out 1029 miles apart, it will take 1029 ÷ 98 = **10.5** hours for them to pass each other.

94. Using a calculator to divide 1 by 27, we see that this is a non-terminating decimal with three digits repeating: 0.037037…. The third digit of the repeating pattern, the 7, will be in the 3rd, 6th, 9th, ..., 36th, 39th and 42nd places after the decimal point. Because the 42nd place contains the digit 7, the place before that, the 41st place, will contain the digit **3**.

95. Since the mode, median and mean form an increasing sequence and we are trying to maximize the value of y, let's start by assuming $x = 11$ and $y \geq 17$. Our list of integers is now, from least to greatest, 11, 11, 13, 15, 17, y. Now these integers have a mode of 11, a median of 14 and a sum of 67 + y. The difference between the median and the mode is 14 − 11 = 3. So in order for the mode, median and mean to form an increasing sequence, the mean must be 14 + 3 = 17. We can write the following equation for the mean: $(67 + y) \div 6 = 17$. Solving for y, we see that $67 + y = 102$, so y = **35**, which is the greatest possible value of y.

96. The semicircle has radius 13/2 = 6.5 and area $(\pi \times 6.5^2) \div 2 = 21.125\pi$ units^2. The area of the triangle is (1/2) × 5 × 12 = 30 units^2. The difference between these two areas, $21.125\pi - 30 \approx$ **36** units^2, is the total area of the shaded regions.

97. Each of the 12 tourists has two choices for a tour. That would be $2^{12} = 4096$ different ways to split into two groups. But we cannot allow all the tourists to go on either one of the tours, so our answer is 4096 − 2 = **4094** ways.

Circles	1	2	3	4	⋯	n
Intersections	0	2	6	12	⋯	$n(n-1)$

98. It's virtually impossible to draw 30 circles carefully enough to count the number of distinct intersections. You might try drawing one circle and then adding more circles, one at a time, and counting intersections to see if a pattern emerges. Doing so reveals that with 1 circle, there are 0 intersections; with 2 circles, there are, at most, 2 intersections; with 3 circles, there are, at most, 6 intersections; with 4 circles, there are, at most, 12 intersections. Organizing these findings in a table like the one shown, we see that for n circles, there appear to be $n(n-1)$ distinct intersections. Therefore, 30 circles will have, at most, 30 × 29 = **870** intersections. Alternatively, since every pair of circles intersects in, at most, 2 points, and there are ${}_{30}C_2$ ways to select a pair from a collection of 30 circles, it follows that there are 2 × (30 ×29)/2 = 30 × 29 = **870** intersections.

99. The sphere has radius r, so its "height" would be $2r$. This means the height of both the cone and the cylinder is $2r$. The volume of the cylinder is $\pi \times r^2 \times h = \pi \times r^2 \times 2r = 2 \times \pi \times r^3$. The volume of the cone is $(1/3) \times \pi \times r^2 \times h = (1/3) \times \pi \times r^2 \times 2r = (2/3) \times \pi \times r^3$. The volume of the sphere is $(4/3) \times \pi \times r^3$. The volumes of the cone and sphere, combined, are $(2/3) \times \pi \times r^3 + (4/3) \times \pi \times r^3 = 2 \times \pi \times r^3$, which is exactly the volume of the cylinder. So **100**% of the cylinder is filled.

100. Let's list all the odd numbers less than 500 that we can create with 3s and 4s and strike through those that are not prime: 3, ~~33~~, 43, ~~333~~, ~~343~~, 433, 443. The sum of the primes among them is 3 + 43 + 433 + 443 = **922**.

Warm-Up 7

101. Mac has 25 × 0.2 = 5 red marbles, and Thayer also has 20 × 0.25 = 5 red marbles. Thus, the absolute difference between the numbers of red marbles they have is **0** marbles.

102. First we evaluate $g(-3) = 2(-3) + 4 = -6 + 4 = -2$. Next we evaluate $f(-2) = (-2)^2 - 2 = 4 - 2 =$ **2**.

103. A cube has 6 faces, 12 edges and 8 vertices. When each vertex is cut off, 1 new equilateral triangle face is created, though the size of the triangle can vary. There is a net gain of, at most, 3 edges for a maximum of 8 × 3 = 24 additional edges. There is also a net gain of, at most, 2 vertices, for a maximum of 8 × 2 = 16 additional vertices. At most, the sum of the number of faces, edges and vertices will be (6 + 8) + (12 + 24) + (8 + 16) = 14 + 36 + 24 = **74** . Note that this maximum can be achieved if the cube is cut so that for each new triangle face none of its vertices reaches the midpoint of an edge of the cube (Fig. 1). Otherwise, the resulting solid would have fewer edges and vertices (Fig. 2).

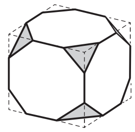


Figure 1

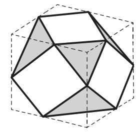


Figure 2

104. Since 4 oranges cost 90 cents, 1 dozen oranges cost 3 × 90 = 270 cents. Therefore, 3 dozen oranges cost 3 × 270 = 810 cents, or $**8.10**.

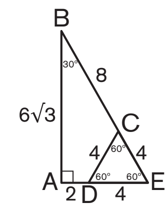
105. Let's imagine that we have 100 pounds of fully matured grapes, consisting of 80 pounds of water and 20 pounds of what we will call pulp. After the drying process, the amount of pulp is still 20 pounds, but the raisins are now only 20% water. The pulp accounts for the other 80%, and the raisins must weigh 25 pounds since 20 is 80% of 25. So, 5 out of 80 pounds of water remain, which is 5/80 = **1/16** of the original water.

106. We are looking for a number with lots of factors, probably a number that is a multiple of both 2 and 3, which makes it a multiple of 6. It turns out that **48** has the greatest sum of proper factors. That sum is 1 + 2 + 3 + 4 + 6 + 8 + 12 + 16 + 24 = 76.

107. We see that $(AAA)^3$ has a units digit of 7, which can only occur if the number being cubed has a units digit of 3, so A = **3**.

108. First, we note that the syrup maker initially has 25 liters of maple syrup and 75 liters of base. She will need to add 50 liters of maple syrup to make the ratio 1:1. Since 90%, or 9/10 of the amount of sap will evaporate, she can obtain those 50 liters of maple syrup from 50 × 10 = **500** liters of maple sap.

109. Suppose we extend sides AD and BC to intersect at E, as shown. Since the $m\angle B = 360 - (90 + 120 + 120) = 360 - 330 = 30$ degrees, it follows that $\triangle ABE$ is a 30-60-90 right triangle. The area of ABCD is the area of $\triangle ABE$ minus the area of $\triangle CDE$. Within the small triangle, since $m\angle C = m\angle D = 180 - 120 = 60$ degrees, we know that $\triangle CDE$ is equilateral, so CD = DE = CE = 4 units. The area of an equilateral triangle with side length s is $(1/4) \times s^2\sqrt{3}$. So $\triangle CDE$ has area $(1/4) \times 16\sqrt{3} = 4\sqrt{3}$ units². For $\triangle ABE$, BE = 8 + 4 = 12 units. In a 30-60-90 right triangle, the length of the short leg is half that of the hypotenuse, and the length of the long leg is $\sqrt{3}$ times the length of the short leg. So AE = 12 ÷ 2 = 6 units, and AB = $6\sqrt{3}$ units. Triangle ABE, then, has area $(1/2) \times 6 \times 6\sqrt{3} = 18\sqrt{3}$ units². Therefore, the area of ABCD is $18\sqrt{3} - 4\sqrt{3} = \mathbf{14\sqrt{3}}$ units².



110. The next 2 five-letter arrangements that follow Zzyzx alphabetically are Zzyzy and Zzyzz. The remaining arrangements are of the form Zzz _ _. There are 26 × 26 = 676 ways to fill in the last two letters, so following Zzyzx alphabetically, there are 2 + 676 = **678** arrangements.

Warm-Up 8

111. We know that $z = 20$. So, $x = 3 \times 20 = 60$. Thus, $60/y = 10$ and $y = 60/10 = \mathbf{6}$.

112. There are $2^5 = 32$ different ways the balls can be distributed between the two containers. Having the two red balls alone in the red container is just one of these 32 possibilities. The probability of this occurring, therefore, is **1/32**.

113. The diameter of the circle, which is the distance from A to B, is $d = \sqrt{[(10 - (-2))^2 + (2 - 4)^2]} = \sqrt{(12^2 + (-2)^2)} = \sqrt{(144 + 4)} = \sqrt{148} = 2\sqrt{37}$ units. The radius of the circle is half this amount, which is $\sqrt{37}$ units, so the area of the circle is $\pi \times r^2 = \pi \times (\sqrt{37})^2 = \mathbf{37\pi}$ units².

114. Since the sequence is arithmetic, the difference of 56 − 20 = 36 must be divided into equal intervals. We can do this using the factors of 36. If we list the factors of 36 in pairs, we can think of the first factor as the number of intervals and the second factor as the size of each interval: 1 × 36, 2 × 18, 3 × 12, 4 × 9, 6 × 6, 9 × 4, 12 × 3, 18 × 2 and 36 × 1. That makes **9** possible sets of integers that form arithmetic sequences.

115. If you don't recall that the interior angle measures of a regular pentagon are 108 degrees, you can subdivide the pentagon into three triangles. Since the sum of the interior angles of a triangle is 180 degrees, the sum of the interior angles of the pentagon must be 3 × 180 = 540 degrees. Divide by 5, and we find that each interior angle measures 108 degrees. The isosceles right triangles have angle measures of 45, 45 and 90. The vertex angle of each shaded triangle is what is left when two 45-degree angles are subtracted from the 108-degree angle, which is **18** degrees.

116. There are 12 × 12 = 144 tiles on the bathroom floor. We can group the tiles into three categories: 4 corner tiles, 40 edge tiles and 100 interior tiles. There is a 100/144 chance that the first of the wrong-colored tiles is placed among the 100 interior tiles. If this happens, there is a 4/143 chance that the other wrong-colored tile shares an edge with it. There is a 40/144 chance that the first wrong-colored tile is one of the 40 tiles around the edge of the room. If this happens, there is a 3/143 chance that the other wrong-colored tile shares an edge with it. Finally, there is a 4/144 chance that the first wrong-colored tile is placed in one of the 4 corners. If this happens, there is a 2/143 chance that the other wrong-colored tile shares an edge with it. The total probability, then, that the two wrong-colored tiles share an edge is (100/144) × (4/143) + (40/144) × (3/143) + (4/144) × (2/143) = 528/(144 × 143) = (11 × 48)/(144 × 143) = **1/39**. Alternatively, there are 11 ways for tiles to share an edge in each row and 11 ways in each column, for a total of 2 × 12 × 11 ways. Dividing this by the number of ways to place the two tiles, which is (144 × 143)/2, we get (2 × 12 × 11) ÷ (144 × 143)/2 = (2 × 12 × 11 × 2)/(144 × 143) = **1/39**.

117. The units digits of the powers of 3 repeat in a cycle of four numbers: 3, 9, 7, 1. Since $17 \equiv 1 \bmod 4$, the units digit of 3^{17} will be the first number in this pattern, which is 3. Similarly, the units digits of the powers of 7 repeat in the pattern 7, 9, 3, 1. Since $23 \equiv 3 \bmod 4$, the units digit of 7^{23} will be the third number in this pattern, which is 3. The units digit of the product $3^{17} \times 7^{23}$ will be the product of the units digits of 3^{17} and 7^{23}, which is 3 × 3 = **9**.

118. The geometric mean of two numbers is the square root of the product of the two numbers. For 14 and 126, the geometric mean is $\sqrt{(14 \times 126)} = \sqrt{(14 \times 14 \times 9)} = \sqrt{(14^2 \times 3^2)} = 14 \times 3 = \mathbf{42}$.

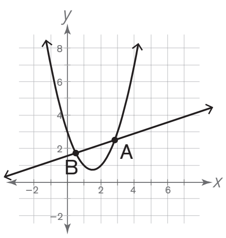
119. The area of the larger circle is $\pi \times 12^2 = 144\pi$ inches^2. The diagonal of the square is the same as the diameter of the circle, which is $2 \times 12 =$ 24 inches. By the properties of 45-45-90 right triangles, we know that the side length of the square is $24/\sqrt{2} = 12\sqrt{2}$, which is also the diameter of the smaller circle. So the radius of the smaller circle is $12\sqrt{2} \div 2 = 6\sqrt{2}$. The area of the smaller circle, then, is $\pi \times (6\sqrt{2})^2 = 72\pi$. The shaded region has area $144\pi - 72\pi =$ **72π** in^2.

120. The base-10 value of 321_5 is $3 \times 5^2 + 2 \times 5^1 + 1 \times 5^0 = 75 + 10 + 1 = 86$. The base-10 value of 321_4 is $3 \times 4^2 + 2 \times 4^1 + 1 \times 4^0 =$ $48 + 8 + 1 = 57$. Their base-10 sum is $86 + 57 = 143$. We can write 143 as a sum of powers of 3 as follows: $143 = \underline{1} \times 3^4 + \underline{2} \times 3^3 + \underline{0} \times 3^2 + \underline{2} \times 3^1 + \underline{2} \times 3^0$. Therefore, the sum in base 3 is **12022**.

Workout 5

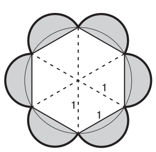
121. If six numbers have a mean of 6, then their sum must be $6 \times 6 = 36$. Since we want to maximize the sum of the largest three numbers, we should minimize the sum of the smallest three numbers. The least possible sum would be $1 + 2 + 3 = 6$. We need to see if this works with the given conditions. With 6 as the median and 12 as the greatest number in the set, we could have 1, 2, 3, 9, 9, 12. But this won't work because the set does not contain six different integers. If we change the third and fourth numbers to 4 and 8, respectively, we have 1, 2, 4, 8, 9, 12. The median is still 6, and there are no duplicates in the set. The sum of the three largest numbers is $8 + 9 + 12 =$ **29**.

122. Because Nathan ran 12 seconds per mile slower than his goal in the first half of the run, it would seem that he should run 12 seconds per mile faster than his goal in the second half. That would be a pace of 7 minutes 12 seconds per mile, or **7.2** minutes per mile. We should confirm this using a different approach. To complete the 5-mile run at an average pace of 7 minutes 24 seconds per mile, or 7.4 minutes per mile, Nathan must finish in $7.4 \times 5 = 37$ minutes. Running at a pace of 7 minutes 36 seconds per mile, or 7.6 minutes per mile, for the first 2.5 miles took $7.6 \times 2.5 =$ 19 minutes. Nathan would then need to complete the final 2.5 miles in $37 - 19 = 18$ minutes. This confirms that his pace should be $18 \div 2.5 =$ **7.2** minutes per mile.



123. Let's get an idea of what the functions $y = x^2 - 3x + 3$ and $4x - 12y = -19$ look like and how they intersect each other by graphing them on a coordinate grid as shown. We have a line that intersects a parabola at two points. Solving the second equation for y yields $4x - 12y = -19 \rightarrow 12y = 4x + 19 \rightarrow y = (4x + 19)/12$. Setting the expressions for y equal to each other and solving for x, we get $x^2 - 3x + 3 = (4x + 19)/12 \rightarrow 12(x^2 - 3x + 3) = 4x + 19 \rightarrow 12x^2 - 36x + 36 =$ $4x + 19 \rightarrow 12x^2 - 40x + 17 = 0$. Factoring the quadratic equation gives us $(2x - 1)(6x - 17) = 0$. So $2x - 1 = 0$ and $x = 1/2$ or $6x - 17 = 0$ and $x = 17/6$. The sum of these x-coordinates is $1/2 + 17/6 = (3 + 17)/6 = 20/6 =$ **10/3**. Alternatively, the sum of the roots of a quadratic equation is always the opposite of the coefficient of the linear term when the quadratic is written with leading coefficient 1. The equation $12x^2 - 40x + 17 = 0$ written in the required form is $x^2 - (10/3)x + (17/12)$. The sum of the roots, then, is the opposite of $-10/3$, which is **10/3**.

124. ABC is a 30-60-90 right triangle. The ratio of the short leg to the long leg of a 30-60-90 triangle is 1 to $\sqrt{3}$, so the length of AC must be $9 \div \sqrt{3} = 3\sqrt{3}$ cm. Triangle ACD is also a 30-60-90 triangle, so the same ratio applies and length DC must be $3\sqrt{3} \div \sqrt{3} =$ **3** cm.



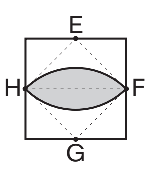
125. The diameter of the circle is $1 \times 2 = 2$ units. If we use the vertices of the hexagon to draw three diameters, we divide the hexagon into 6 equilateral triangles, as shown. The side length of the regular hexagon, also the diameter of each semicircle, then, is 1 unit. When paired, the six semicircles together make three circles of diameter 1 unit. The sum of their circumferences, which is also the perimeter we seek, is $3 \times \pi \times 1 \approx$ **9.4** units.

126. This organized list shows all the three-digit positive integers whose digits have a sum of 9. There are $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 =$ **45** such integers.

108	207	306	405	504	603	702	801	900
117	216	315	414	513	612	711	810	
126	225	324	423	522	621	720		
135	234	333	432	531	630			
144	243	342	441	540				
153	252	351	450					
162	261	360						
171	270							
180								

127. There are $2 \times 2 \times 2 \times 2 = 16$ ways the next four yo-yos can exit the assembly line. In only 2 of these 16 ways do the yo-yos all have the same color (all blue or all red), so the probability is 2/16, or **1/8**.

128. The diameter of the cup is 6 cm, so the radius is 3 cm. The volume of the cup is $\pi \times r^2 \times h = \pi \times 3^2 \times 12 = 108\pi$ cm^3. The 100π mL of tea will occupy 100π cm^3 of space, so the spherical bubbles need to occupy the remaining $108\pi - 100\pi = 8\pi$ cm^3. Each spherical bubble has volume $8\pi \div 48 = \pi/6$ cm^3. So $4/3 \times \pi \times r^3 = \pi/6$. Solving for r, we get $r^3 = (1/6) \times (3/4) \rightarrow r^3 = 1/8 \rightarrow r = \sqrt[3]{(1/8)} =$ **0.5** cm.



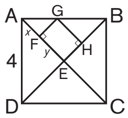
129. Square ABCD has side length 2 m, so the right triangle at each corner has side length 1 m and hypotenuse length $\sqrt{2}$ m. The hypotenuse length is also the radius of the arcs of the circles. Since triangles EFH and GFH are right triangles, it follows that the area of each sector is 1/4 of the area of a circle with radius $\sqrt{2}$ m. As the figure shows, the area of the shaded region is the difference between the combined areas of the sectors, $2(1/4 \times \pi \times (\sqrt{2})^2) = \pi$ m^2, and the combined area of triangles EFH and GFH, $(\sqrt{2})^2 = 2$ m^2. Therefore, the shaded region has area $\pi - 2 \approx$ **1.14** m^2.

130. We should consider the LCMs of all pairs of guesses, as shown in the table. The value of k cannot be 60, because that would make three statements correct. The next possibility is 90, which would make only Bruce and Steven correct. The least possible value of k, then, is **90**.

LCM(15, 18) = 90
LCM(15, 20) = 60
LCM(15, 28) = 420
LCM(15, 60) = 60
LCM(18, 20) = 180
LCM(18, 28) = 252
LCM(18, 60) = 180
LCM(20, 28) = 140
LCM(20, 60) = 60
LCM(28, 60) = 420

Workout 6

131. The consecutive integers from –37 to 37 have a sum of zero, so we should start with 38 and add consecutive integers until we get a sum of 200. Five times 40 is 200, so let's try 38 + 39 + 40 + 41 + 42. This works. The greatest integer in Caynan's sequence is **42**.



132. In the figure, right triangle AFG is isosceles with legs of length x, and rectangle EFGH has sides of lengths x and y. That means that AE = $x + y$. Square ABCD, with side length 4 units, has diagonal length $4\sqrt{2}$ units. Notice that AE is half the diagonal length, so we can write $x + y = 2\sqrt{2} = \sqrt{(2^2 \times 2)} = \sqrt{8}$. Therefore, z = **8**.

133. Two letters can be chosen from the 26 slips of paper in 26 × 25 ÷ 2 = 325 different ways. The 7-letter word ALGEBRA contains 6 different letters, which can be paired in 6 × 5 ÷ 2 = 15 different ways. The probability that two randomly drawn slips are one of those pairs is 15/325 = **3/65**.

134. The radius of the pool is 12 feet = 12 × 12 = 144 inches. Since 4.5 feet = 12 × 4 + 6 = 54 inches, we know that Maxwell will fill the pool to a height of 54 – 3 = 51 inches. So the volume of water used is $\pi \times 144^2 \times 51 = 1{,}057{,}536\pi$ in^3. That's 1,057,536π/231 gallons of water. Since every gallon costs 0.15 cent, this will cost 1,057,536π/231 × 0.0015 ≈ $**22**.

135. The place values for this binary number, from left to right, are 2^7, 2^6, 2^5, 2^4, 2^3, 2^2, 2^1 and 2^0. So $10111010_2 = 1 \times 2^7 + 0 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 128 + 32 + 16 + 8 + 2$. Since 8 is a power of 2, there is no need to find the base 10 value of this number to convert it to base 8. We know that $128 = 2 \times 8^2$, $32 + 16 + 8 = (4 + 2 + 1) \times 8^1 = 7 \times 8^1$ and $2 = 2 \times 8^0$. Therefore, $10111010_2 = 2 \times 8^2 + 7 \times 8^1 + 2 \times 8^0$ = **272** base 8.

136. To raise the water in the tank by 1 inch, the metal spheres will need to occupy 10 × 15 × 1 = 150 in^3 of space. Each sphere has a radius of 1/6 ÷ 2 = 1/12 inch and a volume of $4/3 \times \pi \times (1/12)^3 = \pi/1296$ in^3. To raise the water level 1 inch, it takes 150 ÷ (π/1296) ≈ **61,900** spheres.

137. If Leon's favorite pens were previously x dollars for a box of n pens, then the cost per pen was x/n dollars per pen. A reduction of 10% in the price means that each box now costs $0.9x$. With 25% fewer pens, each box now contains $0.75n$ pens. So the cost per pen is now $0.9x/0.75n = 1.2(x/n)$. That's an increase in cost of 1.2 – 1 = 0.2 = **20**% per pen.

138. This problem is a calculator exploration. When the numbers begin displaying in scientific notation, it helps to recognize that the number of digits is one more than the exponent. Using the calculator, we see that 10! has 7 digits; 15! has 13 digits; 20! has 19 digits; 21! has 20 digits; 22! has 22 digits. This means that 22 is the *least* integer for which n! has n digits. Continuing on, we see that 23! has 23 digits; 24! has 24 digits; 25! has 26 digits. Since the number of digits keeps increasing at least as fast as the number does when we pass 24, it follows that **24** is the greatest integer for which n! has n digits.

1	3	5	15
2	6	10	30
4	12	20	60
8	24	40	120

139. The number of sides on Percy's polygon must be a factor of 120. Once we've listed the factors, it's not difficult to find the sum by using a calculator. If, however, we had to find the sum without the aid of a calculator, we could arrange the factors in an array such as the one shown to find the sum more directly. The sum of the numbers in the first column, 1 + 2 + 4 + 8 = 15, times the sum of the numbers in the first row, 1 + 3 + 5 + 15 = 24, is 15 × 24 = 360, which is the sum of all the factors of 120. Since we cannot have a polygon with fewer than 3 sides, and since Percy's polygon is not a triangle or an octagon, we need to subtract 1 + 2 + 3 + 8 = 14 from that sum. Our answer is 360 – 14 = **346**.

140. On Mars a standard year has 95 weeks and each week has 7 sols, for a total of 95 × 7 = 665 sols. During a leap year, there is one more week, or an additional 7 sols, for a total of 672 sols. Since Mars has a 668-sol year, there are an average of 668 sols per year. If we let n and m represent the number of standard years and leap years, respectively, we have $(665n + 672m) \div (n + m) = 668 \rightarrow 665n + 672m = 668n + 668m \rightarrow 4m = 3n \rightarrow m/n = 3/4$. The ratio of leap years to standard years is 3 to 4. So 3 out of every 7 years, or **3/7** of the years, are leap years.

Warm-Up 9

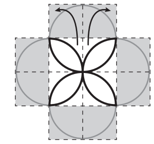
141. There are many pairs of numbers to consider with a sum of 11. However, there are only four factor pairs of 24 to consider: 1 × 24, 2 × 12, 3 × 8 and 4 × 6. The pair 3 and 8 have a sum of 11 and an absolute difference of 8 – 3 = **5**.

142. Varun walked 2/5 of the length of the walkway, so Becca must have also walked 2/5 of the length of the walkway. In the same amount of time, the walkway took her the other 3/5 of the length of the walkway. The ratio of the walkway's speed to Becca and Varun's speed is 3/5 to 2/5 = **3/2**.

143. Suppose the original pentagon has side length 1 unit and perimeter 1 × 5 = 5 units. When two sides of this pentagon are doubled, the perimeter of the new pentagon is 1 × 3 + 2 × 2 = 3 + 4 = 7 units. The increase in the perimeter is (7 – 5)/5 = 2/5 = **40**%.

144. Let d represent the number of dimes Joe has. Joe has 4 more nickels than dimes, so he has $d + 4$ nickels. He has 2 more quarters than nickels, so he has $d + 4 + 2 = d + 6$ quarters. Joe has a total of 37 coins, so we can write the equation $d + d + 4 + d + 6 = 37$. This simplifies to $3d + 10 = 37$. So $3d = 27$, and $d = 9$. If follows, then, that Joe has $d + 6 = 9 + 6$ = **15** quarters.

145. A six-digit repeating integer can take three different forms: a three-digit sequence occurring twice, a two-digit sequence occurring three times or one digit occurring six times. There are $9 \times 10 \times 10 = 900$ ways to choose a three-digit repeating sequence, and there are $9 \times 10 = 90$ ways to choose a two-digit repeating sequence, for a total of $900 + 90 = 990$ repeating integers. But the 9 sequences of one digit occurring six times (111111 to 999999) have been counted twice, which means that $990 - 9 =$ **981** numbers are six-digit repeating integers.



146. In the figure shown, each circle is divided into quarters. The arrows show how the pieces can be rearranged to form four congruent shaded rectangles. Each rectangle is 2 cm by 4 cm, so the total area of the shaded regions is $4 \times (2 \times 4) =$ **32** cm^2.

147. The two-digit representations of all the months are valid days. The two-digit representation of only 12 days represent valid months. Thus, there would be $12 \times 12 = 144$ dates for which both interpretations are valid. But we need to exclude the 12 dates for which the month and day are the same. That leaves $144 - 12 =$ **132** dates.

148. We can set the two expressions equal to one another and solve for x. We have $6x - 6 = 2x + 14 \rightarrow 4x = 20 \rightarrow x = 5$. The square base of the pyramid, then, has side length $2x + 14 = 2 \times 5 + 14 = 10 + 14 = 24$ units and area 576 units^2. The four triangular sides of the pyramid each have side length 24 units, height $12\sqrt{3}$ units and area $(1/2) \times 24 \times 12\sqrt{3} = 144\sqrt{3}$ units^2. The surface area of the pyramid is $4 \times 144\sqrt{3} + 576 = 576\sqrt{3} + 576$, which can be rewritten as $24^2(\sqrt{3} + 1)$. Therefore, $a = 24$, $b = 3$, $c = 1$ and $a + b + c = 24 + 3 + 1 =$ **28**.

149. It's probably safe to assume that D doesn't represent 0, 1, 2, 5, 7 or 8 since those digits are shown in the equation. That leaves 3, 4, 6 and 9 as the possible digit that D represents. Since there are only four possibilities, we could start trying them to see which one makes the equation true. If D represents 3, we get $23 \times 351 = 8073$. If D represents 4, 6 or 9, the equation is not true. Therefore, D must represent the digit **3**.

150. Using only the digits 2 through 9, the number of unique three-digit extensions that can be assigned is $8 \times 8 \times 8 =$ **512** extensions.

Warm-Up 10

151. The rectangular prism should be as close to a cube as possible, given the available factors. The prime factorization of 2016 is $2^5 \times 3^2 \times 7$. We want to assign these prime factors to three groups that have as close to the same product as possible. One triple with a product of 2016 is $8 \times 14 \times 18$, which makes the sum of the dimensions $8 + 14 + 18 = 40$. Another triple is $9 \times 14 \times 16$, for which $9 + 14 + 16 = 39$. The triple with the lowest sum, however, is $12 \times 12 \times 14$, with a sum of $12 + 12 + 14 =$ **38**.

152. There are two pairs of digits that have to have a sum of 10, and the sum of all the digits has to be a multiple of 10. Since there are only five digits, this means that one of the digits has to be a zero. Also, since the leftmost digit of the number is 7 and we need two pairs of adjacent numbers that differ by 1, it makes sense for the zero to be the rightmost digit. For the first and second digits to differ by 1, the second digit needs to be 6 or 8. The 8 is more promising since it is also a multiple of 4 and allows us to put 2, which is $8 \div 4$, in the third place. So far we have 782_0. The missing digit adjacent to the 2 needs to be 1 or 3. The sum of that digit and 7 has to be 10, so the missing digit is 3, and the ZIP code is **78230**.

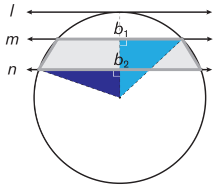
153. We might consider trying to find the prime factorization of 140,209, but it's not divisible by 2, 3, 5, 7 or 11, so this approach could take a while. (It turns out that ABC and CBA are the two prime factors of 140,209.) Another approach is to consider the units digit 9. Only 1×9, 3×3 and 7×7 yield a units digit of 9. In our problem, A, B and C must each represent a different digit, so this means A and C can only be 1 and 9. At this point, we can just try different values of B, and it turns out that $149 \times 941 = 140{,}209$. The value of A + B + C is $1 + 4 + 9 =$ **14**.

154. Since all the triangles are similar, corresponding angles are congruent. So we can let x represent the measure of each of the base angles of each triangle. Then the vertex angle has degree measure $(x + x) \times 2 = 4x$. The sum of the measures of the angles in a triangle is 180 degrees. We have $x + x + 4x = 180$, so $6x = 180$ and $x = 30$. We can see that 8 of these base angles come together at the center of the fan, leaving a total of $360 - 8 \times 30 = 360 - 240 = 120$ degrees for the congruent angles in the spaces between the blades. Angle α measures $120 \div 4 =$ **30** degrees.

155. The twelve couples amount to 24 participants. So when forming a team of three people, there are 24 choices for the first team member. Since the individual with whom the first team member is coupled is no longer an option for this team, there are 22 choices for the second team member. Again, the individuals with whom the first two team members are coupled cannot be part of this team. That means there are 20 choices for the third team member, which gives us a total of $24 \times 22 \times 20 = 10{,}560$ ways to form a team of three. But this includes the 6 orders in which each team can be assembled. Therefore, the number of different teams of three that can be formed is $10{,}560 \div 6 =$ **1760** teams.

156. The algebra below shows one way to find the fraction that is equivalent to the repeating decimal. Let x represent the common fraction equivalent to $0.3\overline{27}$. Then $10x = 3.\overline{27}$ and $1000x = 327.\overline{27}$. Subtracting these two equations yields the following:

$$\begin{aligned} 1000x &= 327.\overline{27} \\ \underline{-10x} &\underline{= \quad 3.\overline{27}} \\ 990x &= 324 \\ x &= \frac{324}{990} = \mathbf{\frac{18}{55}}. \end{aligned}$$



157. The area of a trapezoid with height h and bases of length b_1 and b_2 is $1/2 \times h \times (b_1 + b_2)$. The figure shows the trapezoid in the circle. To find b_2, we can construct the right triangle shown on the left in the circle. This triangle has a hypotenuse of length 3 cm and a short leg of length 1 cm. The long leg, then, has length $\sqrt{(3^2 - 1^2)} = \sqrt{8}$ cm. So, $b_2 = 2\sqrt{8}$ cm. To find b_1, we can construct the right triangle shown on the right in the circle. This triangle has a hypotenuse of length 3 cm and a short leg of length 2 cm. The long leg, then, has length $\sqrt{(3^2 - 2^2)} = \sqrt{5}$ cm. So, $b_1 = 2\sqrt{5}$ cm. We know that the distance between the bases is 1 cm, so the area of the trapezoid is $1/2 \times h \times (b_1 + b_2) = 1/2 \times 1 \times (2\sqrt{8} + 2\sqrt{5}) = \sqrt{8} + \sqrt{5}$ cm^2. Therefore, $ab = 8 \times 5 =$ **40**.

158. Let $n = 2017$ and substitute to get the following: $(n^2 + 11n - 42)/(n - 3)$. Now we can simplify the numerator by factoring. For the fraction, we get $(n + 14)(n - 3)/(n - 3)$, which simplifies to $n + 14$. Substituting 2017 for n yields the answer: $n + 14 = 2017 + 14 =$ **2031**.

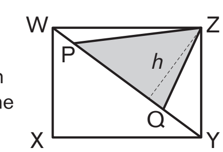
159. We need to know how many 2s we can factor out of $32! = 32 \times 31 \times 30 \times \ldots \times 3 \times 2 \times 1$. We can factor a 2 out of each of the 16 multiples of 2 from 1 to 32; we can factor another 2 out of each of the 8 multiples of 4; we can factor another 2 out of each multiple of 8, and so forth. There are $16 + 8 + 4 + 2 + 1 = 31$ factors of 2, so $n =$ **31**.

160. First, we should note that the total value of the coins will be less than \$1 if all the allowed 3 quarters are not used. We now have an easier problem to solve: how many combinations of dimes, nickels and pennies are there with a total value of at least 25¢, using no more than three of any single denomination? If all 3 dimes are used, then combining them with any combination of 0, 1, 2 or 3 nickels and 0, 1, 2 or 3 pennies gives us a total value greater than \$1, and that's $4 \times 4 = \underline{16}$ combinations. If 2 dimes are used, then there must be at least 1 nickel, and combined with any combination of 0, 1 or 2 of the remaining nickels and 0, 1, 2 or 3 pennies, the total value is greater than or equal to \$1, for another $3 \times 4 = \underline{12}$ combinations. If 1 dime is used, then all 3 nickels must be used. Adding to that, 0, 1, 2 or 3 pennies yields another $\underline{4}$ combinations. In all, there are $16 + 12 + 4 =$ **32** combinations.

Warm-Up 11

161. There are $2 \times 2 \times 2 = 8$ possible outcomes when three coins are flipped. The outcome of 0 tails can occur 1 way and has a probability of 1/8. The outcome of 1 tail can occur 3 ways, so the probability of this outcome is 3/8. The outcome of 2 tails can also occur 3 ways, so the probability of this outcome is also 3/8. The outcome of 3 tails can occur 1 way and has a probability of 1/8. So, the probability that all three students get 0 tails is $1/8 \times 1/8 \times 1/8 = 1/512$, as is the probability that all three get 3 tails. The probability that all three get 1 tail is $3/8 \times 3/8 \times 3/8 = 27/512$, as is the probability that all three get 2 tails. Therefore, the probability that all three get the same number of tails is $(1 + 1 + 27 + 27)/512 = 56/512 =$ **7/64**.

162. The earliest possible sum date is 01/01/02, but the year 2002 will have *only* this one sum date. When determining the earliest year with 12 sum dates, we have to make sure that December has a sum date as well. This doesn't happen until 12/01/13. The latest possible sum date is 12/31/43, but this is again the *only* sum date for the year. When finding the latest year with 12 sum dates, we have to make sure that January and February have sum dates. The year 2032 works for January (01/31/32), but February 29th for this leap year will only give us $2 + 29 = 31$. The year 2031 is not a leap year, so February will not have a sum date. We have to go back to 2030 for February to give us $2 + 28 = 30$. This means the years from 2013 to 2030 all have 12 sum dates, one for each month. That's a total of $2030 - 2012 =$ **18** years.

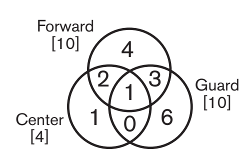


163. Since $3(WP + QY) = 2PQ$, we know that $(WP + QY)/PQ = 2/3$ and that $PQ = 3/5 \times WY$. Diagonal WY divides rectangle WXYZ into two congruent right triangles, each with area equal to half that of WXYZ, or $90 \div 2 = 45$ cm^2. We can write the following equation for the area of $\triangle WYZ$: $1/2 \times WY \times h = 45$, where h is the altitude from Z to WY, as shown. The area of $\triangle PQZ$, then, is $1/2 \times PQ \times h = 1/2 \times 3/5 \times WY \times h = 3/5 \times 45 = 3 \times 9 =$ **27** cm^2.

164. Starting at the point number 1 and drawing segments between points that are 8 spaces apart, we see that the 23rd segment drawn will be to point number $(1 + 23 \times 8) \bmod 50 = 185 \bmod 50 = 35$. The other endpoint of the segment will be point number $35 - 8 = 27$. The sum of these numbers is $27 + 35 =$ **62**.

165. If the figure could not be rotated, there would be $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways to assign the colors. Each distinct coloring can be rotated 0, 90, 180 or 270 degrees, which makes the number of distinct colorings $120 \div 4 =$ **30** colorings.

166. Suppose the number is ABBA. Its value is 1001A + 110B. Because 1001 is divisible by 7 but 110 is not, B must be divisible by 7, which means that it must be 0 or 7. If B = 7, then the only way for A77A to be divisible by 8 is for A to be 6, and the four-digit number 6776 is a palindrome divisible by both 7 and 8. Otherwise B = 0, and the only way for A00A to be divisible by 8 is for A to be 8, and the four-digit number 8008 is a palindrome divisible by both 7 and 8. The larger of the two palindromes that meet the conditions is **8008**.



167. The Venn diagram shows how the 17 players tried out for the different positions. We started with the 1 player who tried out for all three positions and then filled in the numbers for the players who tried out for exactly two positions. There must have been **6** players who tried out only for guard.

168. We can ignore the area of the base of the cylinder, because the height of the water will drop in proportion with the drop in volume. The height will be $100\% - 8\% = 92\%$ of the original 10 cm, so it will be $0.92 \times 10 =$ **9.2** cm.

169. The Triangle Inequality tells us that in a triangle, the longest side length is always less than the sum of the other two side lengths. In other words, the following three inequalities must be true: $2x + 3x + 7 > 6x - 5$, $2x + 6x - 5 > 3x + 7$, $3x + 7 + 6x - 5 > 2x$. Simplifying each inequality yields the following: $x < 12$, $x > 12/5$, $x > -2/7$. So, $12/5 < x < 12$. Therefore, 3, 4, 5, 6, 7, 8, 9, 10 and 11 are the **9** integer values of x for which a triangle exists with the resulting side lengths.

x	y	$(x+y)^2$
−10	1	81
−5	2	9
−2	5	9
−1	10	81
1	−10	81
2	−5	9
5	−2	9
10	−1	81

170. If we expand $(x + y)^2 = 9$, we get $x^2 + 2xy + y^2 = 9$. Since we are told that $x^2 + y^2 = 29$, we can substitute to get $2xy + 29 = 9$. So $2xy = -20$ and $xy = -10$. The table shows the values of $(x + y)^2$ for ordered pairs of integers whose product is -10. Notice that though there are four ordered pairs for which $(x + y)^2 = 9$, only two have $x > y$, and the smallest value of x is **2**.

Workout 7

171. The curved part of the perimeter of each sector of pizza is $C/8$, and $C = 2 \times \pi \times r$. So, the curved part of the perimeter is $C/8 = (2 \times \pi \times r)/8 = (\pi \times r)/4$. The total perimeter of each sector is $r + r + (\pi \times r)/4 = (8r + \pi r)/4 = r(8 + \pi)/4$. Since this perimeter is 10 inches, the value of r must be $10 \div (8 + \pi)/4 = 10 \times 4/(8 + \pi) \approx 3.59$ inches. Now to find the area of the pizza, we square the radius and multiply by π, which is $3.59^2 \times \pi \approx$ **40.5** in^2.

172. Let's say that Alana usually travels at r mi/h and takes t hours to drive the 18 miles to work. Distance = rate × time, so we have $18 = rt$. With the traffic, she averaged $r - 9$ mi/h and took $t + 1/15$ hour. Since the same distance was covered, we can set two expressions for the distance equal to each other to get $rt = (r - 9)(t + 1/15) \rightarrow rt = rt - 9t + r/15 - 9/15 \rightarrow 9t + 9/15 = r/15 \rightarrow 135t + 9 = r$. Now we can substitute this for r in the equation $18 = rt$ to get $18 = (135t + 9)t \rightarrow 18 = 135t^2 + 9t \rightarrow 0 = 15t^2 + t - 2 \rightarrow 0 = (5t + 2)(3t - 1)$. So $5t + 2 = 0$ and $t = -2/5$, or $3t - 1 = 0$ and $t = 1/3$. The measure of time cannot be a negative number, which means it would normally take Alana 1/3 hour = 20 minutes to get to work. On this particular day with the traffic jam, it took her 20 + 4 = **24** minutes.

173. First we convert 128 miles per hour to feet per second as follows: $\dfrac{128 \text{ } \cancel{\text{miles}}}{1 \text{ } \cancel{\text{hour}}} \times \dfrac{5280 \text{ feet}}{1 \text{ } \cancel{\text{mile}}} \times \dfrac{1 \text{ } \cancel{\text{hour}}}{60 \text{ } \cancel{\text{minutes}}} \times \dfrac{1 \text{ } \cancel{\text{minute}}}{60 \text{ seconds}} =$ $\dfrac{128 \times 5280 \text{ feet}}{60 \times 60 \text{ seconds}} = \dfrac{675{,}840 \text{ feet}}{3600 \text{ seconds}}$. It takes 3.5 seconds to reach this speed, so the acceleration must be $675{,}840/3600 \times 1/3.5 =$ $675{,}840/12{,}600$ feet/second^2. This is faster than the acceleration due to gravity. In fact, it is about $675{,}840/12{,}600 \div 32 \approx$ **1.7** g_n.

174. Half of all positive integers are multiples of 2, but we need to focus on the other half, which are not multiples of 2. Of these numbers, one-third are multiples of 3 and the other two-thirds are not. Putting these two ideas together, we can state that $1/2 \times 2/3 = 1/3$ of positive integers are not divisible by 2 or 3. Similar reasoning suggests that we keep the 4/5 of these numbers that are not divisible by 5. We don't need to check for multiples of 4 since they are multiples of 2, and multiples of 6, since they are multiples of 2 and 3, which we have already considered. Therefore, $1/2 \times 2/3 \times 4/5 = 4/15 \approx$ **27**% of the positive integers are not multiples of 2, 3, 4, 5 or 6.

175. Let's use the method of subtracting the probability of the complement of the desired outcome from 1, since it's easier to calculate the probability that no ball is selected more than once. Thus, the probability that at least one ball is selected more than once is $1 - 75/75 \times 74/75 \times 73/75 \times \cdots \times 57/75 \times 56/75 \approx$ **0.94**.

176. The probability that they both get a hit is $8/20 \times 6/16 =$ **3/20**.

177. If we let the width of the flag be 2 units, then the semicircle has radius 1 unit, and based on properties of 30-60-90 right triangles, the triangle has height $\sqrt{3}$ units. Therefore, the flag has a length of $1 + \sqrt{3}$ units and a total area of $2 \times (1 + \sqrt{3}) = 2 + 2\sqrt{3}$ units^2. The area of the semicircle is $1/2 \times \pi \times r^2 = 1/2 \times \pi \times 1^2 = \pi/2$ units^2. The area of the triangle is $1/2 \times b \times h = 1/2 \times 2 \times \sqrt{3} = \sqrt{3}$ units^2. Therefore, the painted area covers $100\% \times (\pi/2 + \sqrt{3})/(2 + 2\sqrt{3}) \approx$ **60**%.

178. Using units cancellation, we get $\dfrac{100 \text{ } \cancel{\text{bars}}}{11.53 \text{ } \cancel{\text{ticks}}} \times \dfrac{4 \text{ meters}}{5 \text{ } \cancel{\text{bars}}} \times \dfrac{100 \text{ } \cancel{\text{ticks}}}{1 \text{ } \cancel{\text{minute}}} \times \dfrac{1 \text{ } \cancel{\text{minute}}}{60 \text{ seconds}} = \dfrac{100 \times 4 \times 100 \text{ meters}}{11.53 \times 5 \times 60 \text{ seconds}} = \dfrac{40{,}000 \text{ meters}}{3459 \text{ seconds}}$.

Usain Bolt ran 100 meters in 9.58 seconds. So, if $(100/9.58)k = 40{,}000/3459$, then $k = 40{,}000/3459 \times 9.58/100 \approx$ **1.11**.

179. To find the equation of Jason's line, we will use the slope $-1/3 = m$ and the point $(5, -2) = (x, y)$. Substituting these values into the point-slope form $y - y_1 = m(x - x_1)$, we see that the equation for Jason's line is $y - (-2) = (-1/3)(x - 5) \rightarrow y + 2 = (-1/3)x + 5/3 \rightarrow y = (-1/3)x - 1/3$. Amisha's line is perpendicular to Jason's, so its slope is the negative reciprocal of $-1/3$, which is 3. We can find the equation of Amisha's line by using the slope $3 = m$ and the point $(4, 1) = (x, y)$. Again, substituting these values into the point-slope form, we see that the equation of Amisha's line is $y - 1 = 3(x - 4) \rightarrow y - 1 = 3x - 12 \rightarrow y = 3x - 11$. Now we can set these two expressions for y equal to each other to get $(-1/3)x - 1/3 = 3x - 11 \rightarrow -1x - 1 = 9x - 33 \rightarrow 32 = 10x \rightarrow x = 32/10 = 16/5$. Now, let's substitute this value for x in Amisha's equation to get the value of the y-coordinate. We have $y = 3 \times 16/5 - 11 = 48/5 - 11 = (48 - 55)/5 = -7/5$. The sum $x + y$ is $16/5 + (-7/5) =$ **9/5**.

180. At each vertex of the triangle, the sector of the circle is 1/6 of a full circle. The three sectors have a combined area equivalent to $3/6 = 1/2$ of a circle with a radius of 4 units. That area is $1/2 \times \pi \times r^2 = 1/2 \times \pi \times 4^2 = 8\pi$ units^2. The triangle has side length equal to twice the radius of a circle, or 8 units, and its altitude is $4\sqrt{3}$ units. Thus, the triangle has area $1/2 \times b \times h = 1/2 \times 8 \times 4\sqrt{3} = 16\sqrt{3}$ units^2. The shaded region within the triangle but outside the circles has area $16\sqrt{3} - 8\pi \approx$ **2.6** units^2.

Workout 8

181. Triangle XYV is $1/2 \times 1/2 = 1/4$ of the area of square WXYZ. The area of the square must be $4 \times 4/5 =$ **16/5** units^2.

182. We know there is at least one of each denomination, which amounts to $5 + 11 + 19 = 35$ points. That leaves $56 - 35 = 21$ points to account for. There cannot be another 19-point token. The remaining 21 points can only include two 5-point tokens and one 11-point token. To verify, we have $3 \times 5 + 2 \times 11 + 1 \times 19 = 15 + 22 + 19 = 56$ points and a total of $3 + 2 + 1 =$ **6** tokens.

183. Each round of this "game" consists of Xera rolling a die followed by Yeta picking a card, with replacement. In any round, Yeta can win only after Xera first rolls a number other than four. The probability that Yeta wins in any round, then, is $(5/6) \times (1/13) = 5/78$. It is possible for there to be no winner in a round, but we are interested in the probability that when there is a winner, the winner is Yeta. Since the probability that there is no winner in a round is $(5/6) \times (12/13) = 60/78$, it follows that $1 - (60/78) = 18/78$ is the probability that there is a winner in a round. Therefore, the probability that Yeta will win the game is the $(5/78)/(18/78) =$ **5/18**.

184. The volume of a cone is $1/3 \times \pi \times r^2 \times h$. Notice that we multiply by the height h and the square of the radius r. This is why doubling the radius can compensate for reducing the height by one-fourth. We get the same volume because $2^2 \times 1/4 = 4/4 = 1$. If we decrease the height to 1/3 of what it was, the radius has to increase by a number that when squared is 3. That number is $\sqrt{3}$. Since $\sqrt{3} \approx 1.73$, the radius must increase by a factor of $1.73 - 1 = 0.73$, or **73**%.

185. The expression $(n^2 + 9)/(n^2 + 4)$ can be factored to get $((n - 3)(n + 3))/((n - 2)(n + 2))$. Since $n - 3$ and $n - 2$, as well as $n + 3$ and $n + 2$, each differ by 1, $(n - 3)/(n - 2)$ and $(n + 3)/(n + 2)$ are both common fractions. We need to look at $(n - 3)/(n + 2)$ and $(n + 3)/(n - 2)$ to see when either of these is not a common fraction, which will also make the expression $((n - 3)(n + 3))/((n - 2)(n + 2))$ not a common fraction. The pair $n - 3$ and $n + 2$, as well as $n + 3$ and $n - 2$, each differ by 5. When the value of $n - 3$ is a multiple of 5, the value of $n + 2$ will also be a multiple of 5, as is the case with $n + 3$ and $n - 2$. So when any of these factors is a multiple of 5, there will be a multiple of 5 in both the numerator and denominator, and the expression will not be a common fraction. Each multiple of 5 in our range can be assigned to the pair $n - 3$ and $n + 2$, as well as the pair $n + 3$ and $n - 2$. The smallest multiple of 5 we consider is $1 \times 5 = 5$ and the largest is $403 \times 5 = 2015$. So we have 403 integers that can be used twice, and the number of integers n from 1 to 2016 that yield a GCF greater than 1 for the numerator and denominator is $403 \times 2 =$ **806**.

186. In ascending order, the five known numbers are 4, 5, 5, 6 and 7. In an ordered list of six numbers, the median is the mean of the third and fourth values. If $x \le 5$, then the third and fourth values will have a mean of 5. If $x \ge 6$, then the mean of the third and fourth values is $(5 + 6)/2 = 5.5$. Since there are only two possible values for the median, let's see if there are two values of x that will give us the same overall mean and mean of modes as those two values. The sum of the five known numbers is $4 + 5 + 5 + 6 + 7 = 27$. In order for the mean of the six values to be 5, the sum of all six values would have to be $6 \times 5 = 30$, so $x = 3$. The six values 3, 4, 5, 5, 6 and 7 have a mode of 5, so the mean, median and mean of modes are all 5 when $x = 3$. Similarly, in order for the six numbers to have a mean of 5.5, their sum must be $5.5 \times 6 = 33$, so $x = 6$. The six values 4, 5, 5, 6, 6 and 7 have modes 5 and 6, making the mean of the modes $(5 + 6)/2 = 5.5$. So the mean, median and mean of the modes are all 5.5 when $x = 6$. Therefore, 3 and 6 are the **2** possible values of x.

187. There are 13 possible sets of three consecutive numbers in the range 1 to 15. The first starts with 1 and the last starts with 13. There are $15 \times 14 \times 13$ ways to randomly draw three of the slips. The probability is, thus, $13/(15 \times 14 \times 13) = 1/(15 \times 14) =$ **1/210**.

188. Construct a rectangle with length a inches, width b inches and diagonal 8 inches. It has area $a \times b = 26$ in^2 and perimeter $2a + 2b = 2(a + b)$. We have a right triangle with legs a and b inches and hypotenuse 8 inches. Using the Pythagorean Theorem, we can write $a^2 + b^2 = 8^2$. Recall that $(a + b)^2 = a^2 + 2ab + b^2$, which can be rewritten as $a^2 + b^2 + 2ab$. Substituting $8^2 = 64$ for $a^2 + b^2$ and 26 for ab, we get $a^2 + b^2 + 2ab = 64 + 2 \times 26 = 64 + 52 = 116 = (a + b)^2$ and $a + b = \sqrt{116} = 2\sqrt{29}$. The perimeter of the rectangle, then, is $2(a + b) = 2 \times 2\sqrt{29} =$ $\mathbf{4\sqrt{29}}$ inches.

189. There are four similar triangles in the figure, so we will be able to use proportional reasoning to find some of the unknown lengths. Triangles CDF and ECF are similar, so if we let CF = x, we can write and solve the following proportion: $x/8 = 26/x \to x^2 = 8 \times 26 \to x^2 = 208 \to x = \sqrt{208} = 4\sqrt{13}$ units. If we let CE = y, we can use the Pythagorean Theorem with triangle CEF as follows: $8^2 + y^2 = (\sqrt{208})^2 \to y^2 = (\sqrt{208})^2 - 8^2 \to y^2 = 144 \to y = 12$. Now, using similar triangles ECF and BCA, we can find the length of side BC. If we let BC = z, we can write and solve the following proportion: $4\sqrt{13}/12 = 26/z \to 4\sqrt{13} \times z = 12 \times 26 \to z = (12 \times 26)/(4\sqrt{13}) \times (4\sqrt{13})/(4\sqrt{13}) = (312 \times 4\sqrt{13})/(16 \times 13) =$ $\mathbf{6\sqrt{13}}$ units.

190. Since Pump P takes 12 hours to fill the tank, it must fill 1/12 of the tank each hour. Similarly, since Pump Q takes 15 hours, it must fill 1/15 of the tank each hour. Together the two pumps fill $1/12 + 1/15 = 5/60 + 4/60 = 9/60 = 3/20$ of the tank each hour. At this rate, it will take 20/3 hours to fill the tank. When Pump P was turned off, 60% of the total time, or $0.6 \times 20/3 = 4$ hours had elapsed. To fill the remaining 40% of the tank, Pump Q needed another $0.4 \times 15 = 6$ hours. It must have taken a total of $4 + 6 =$ **10** hours to completely fill the tank.

Warm-Up 12

191. From left to right, we'll label the four spaces on the clock (two for the hour and two for the minutes) 1, 2, 3 and 4 so they are easily referenced in this solution. For 5 minutes every hour, there is a 0 in the 4th space of the clock's display when the 3rd space does not show 0 (:10, :20, :30, :40 and :50). For the first 10 minutes of every hour, there is a 0 in the 3rd space of the clock's display (:00 to :09). Not including the hour from 10:00 to 10:59, which we will consider separately, that's $10 + 5 = 15$ minutes every hour for 11 hours a day, which is $15 \times 11 = 165$ minutes. From 10:00 to 10:59, there is a 0 in the 2nd space of the clock's display for all 60 minutes (10:). There is never a 0 in the 1st space of the clock's display. Our total is $165 + 60 =$ **225** minutes.

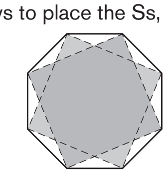
192. Rather than try to subtract with repeating decimals, we should convert each of these numbers to fractions. If we let $x = 1.\overline{18}$ and $y = 2.3\overline{6}$, we can convert these repeating decimals as follows:

$$\begin{array}{rl} 100x = & 118.\overline{18} \\ -\quad x = & 1.\overline{18} \\ \hline 99x = & 117 \\ x = & \frac{117}{99} = \frac{13}{11} \end{array} \qquad \begin{array}{rl} 100y = & 236.\overline{6} \\ -10y = & 23.\overline{6} \\ \hline 90y = & 213 \\ y = & \frac{213}{90} = \frac{71}{30} \end{array}$$

Therefore, $\left|1.\overline{18} - 2.3\overline{6}\right| = \left|\frac{13}{11} - \frac{71}{30}\right| = \left|\frac{13 \times 30 - 71 \times 11}{11 \times 30}\right| = \left|\frac{390 - 781}{330}\right| = \mathbf{\frac{391}{330}}$.

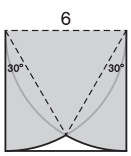
193. It should be noted that this dissection and rearrangement of one square into two other squares is essentially a proof of the Pythagorean Theorem. The area of square DEFP' is 1 × 1 = 1 in^2. Square FPBE' is said to have twice the area, so it has area 2 in^2 and side length $\sqrt{2}$ inches. It follows, then, that the original square has area 1 + 2 = 3 in^2 and side length $\mathbf{\sqrt{3}}$ inches.

194. There would be 8! arrangements of an eight-letter word if all the letters were different. Since the three identical Ss in TRESPASS can be rearranged in 3! = 3 × 2 × 1 ways, we have to divide 8!/3! = 8 × 7 × 6 × 5 × 4 = 6720. Thus, there are 6720 arrangements of the letters in TRESPASS. Since 5 of the 8 letters are not S, there must be 5/8 × 6720 = **4200** arrangements that do not have S as the final letter. (This is actually true for any position in the word.) Alternatively, there are ${}_7C_3$ = 7!/(4! × 3!) = (7 × 6 × 5)/(3 × 2 × 1) = 210/6 = 35 ways to place the Ss, and there are 5! = 120 ways to permute the remaining letters. Again, there are 35 × 120 = **4200** arrangements.



195. There are ${}_8C_4$ = (8 × 7 × 6 × 5) ÷ (4 × 3 × 2 × 1) = 70 different ways to choose 4 of the 8 vertices in a regular octagon. Only 2 of these ways will be the vertices of a square, as shown. So the probability is 2/70 = **1/35**.

196. If we multiply the plane's speed on the return trip by the time it took, we find that the distance traveled was 450 × 2.5 = 1125 miles. To travel this same distance took 3 hours on the first trip, so the average speed was 1125 ÷ 3 = 375 mi/h. If we let r be the rate of the plane and w be the rate of the wind, then for the first trip we have $r - w = 375$ and for the return trip we have $r + 2w = 450$. Solving this system of equations yields $(r + 2w) - (r - w) = 450 - 375 \rightarrow 3w = 75 \rightarrow w = 25$, so the speed of the original headwind was **25** mi/h.

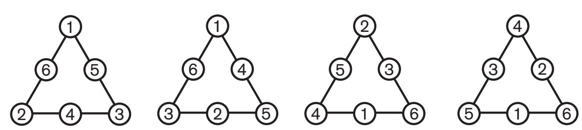


197. The figure shows how we can construct an equilateral triangle from three radii of the quarter-circles. Since the area of an equilateral triangle with side length s is $s^2\sqrt{3}/4$, this triangle of side length 6 cm has area $6^2\sqrt{3}/4 = 36\sqrt{3}/4 = 9\sqrt{3}$ cm^2. The remaining sectors each have a combined degree measure of 30 + 30 = 60, so their combined area is 1/6 the area of a circle of radius 6 cm, or $1/6 \times \pi \times r^2 = 1/6 \times \pi \times 6^2 = 6\pi$ cm^2. The total area of the shaded region, then, is $6\pi + 9\sqrt{3}$ cm^2, so the value of $a + b + c$ is 6 + 9 + 3 = **18**.

198. We need to write 60 as a product of three factors representing the prism's length l, width w and height h such that $l \geq w \geq h$. We have the following organized list of dimensions: 60 × 1 × 1, 30 × 2 × 1, 20 × 3 × 1, 15 × 4 × 1, 15 × 2 × 2, 12 × 5 × 1, 10 × 6 × 1, 10 × 3 × 2, 6 × 5 × 2 and 5 × 4 × 3. These are the dimensions for the **10** prisms that are possible.

199. There are ${}_6C_3$ = 20 ways to select the three numbers for the circles along a side of the triangle. For each of the six circles to be filled with a different number from 1 to 6, the largest sum that can include the number 1 is 1 + 5 + 6 = 12, and the smallest sum that can include the number 6 is 1 + 2 + 6 = 9. If the sums of the numbers along all three sides are to be equal, these sums must range from 9 to 12. Each of the sums 9, 10, 11 and 12 can be obtained in three ways as follows:

9 = 1 + 2 + 6 = 1 + 3 + 5 = 2 + 3 + 4
10 = 1 + 3 + 6 = 1 + 4 + 5 = 2 + 3 + 5
11 = 1 + 4 + 6 = 2 + 3 + 6 = 2 + 4 + 5
12 = 1 + 5 + 6 = 2 + 4 + 6 = 3 + 4 + 5



The figure shows filled-in figures for these **4** solutions.

200. Recall that the infinite series 1/2 + 1/4 + 1/8 + 1/16 + … converges to 1. Thus, the time it takes to run 1 length of the field is $t = 1 + 2/3 + (2/3)^2 + (2/3)^3 + \cdots$. If we multiply both sides of the equation by 3/2, we get $(3/2)t = 3/2 + 1 + 2/3 + (2/3)^2 + (2/3)^3 + \cdots$. Our infinite series appears again. Substituting, we get $(3/2)t = 3/2 + t \rightarrow (1/2)t = 3/2 \rightarrow t =$ **3** minutes.

Warm-Up 13

201. We can factor the difference of cubes to be $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. We know the values of $a - b$ and $a^2 + b^2$. To find ab, let's square both sides of the first equation to get $a - b = 3 \rightarrow a^2 - 2ab + b^2 = 9$. Substituting 65 for $a^2 + b^2$, we get $65 - 2ab = 9 \rightarrow 2ab = 56 \rightarrow ab = 28$. We now have the values for $a^2 + b^2$ and ab. Substituting in the factorization of $a^3 - b^3$, we see that $a^3 - b^3$ is 3 × (65 + 28) = 3 × 93 = **279**.

202. To solve this problem, we will consider three cases.

111	222	333	444
555	666	777	888
999			
102	204	306	408
123	135	147	159
234	246	258	
345	357	369	
456	468		
567	579		
678			
789			

<u>Case I:</u> The first case includes numbers with all three digits the same, of which there are <u>9</u>.
<u>Case II:</u> The next case includes numbers with 0 as one of the three digits. The digits in each of these 4 numbers can be arranged to form 4 different numbers that meet our criteria, for a total of 4 × 4 = <u>16</u>.
<u>Case III:</u> The final case includes numbers with three distinct digits from 1 to 9. The digits in each of these 16 numbers can be arranged to form 6 different numbers that meet our criteria, for a total of 16 × 6 = <u>96</u>.

Altogether, that's 9 + 16 + 96 = **121** integers.

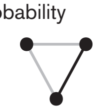
203. There must be more green marbles than red marbles since the probability is greater that the two marbles will be green. Since there are at least two marbles of each color, the smallest case we can try is 3 green marbles and 2 red marbles. The probability that the marbles are both green is 3/5 × 2/4 = 3/10 and the probability that they are both red is 2/5 × 1/4 = 1/10. This makes 2 reds one-third as likely as 2 greens. Increasing the number of green marbles with just 2 red marbles makes the probability of 2 reds even less likely than 2 greens. So, let's try 4 green and 3 red marbles. The probabilities are p(2 greens) = 4/7 × 3/6 = 2/7 and p(2 reds) = 3/7 × 2/6 = 1/7. This has 2 reds half as likely as 2 greens. The minimum number of marbles is **7**. Some students may work this out as a combinatorics question: If you have 4 greens and 3 reds, then you have "4 choose 2" = 4 × 3 ÷ 2 = 6 ways to choose 2 greens and "3 choose 2" = 3 × 2 ÷ 2 = 3 ways to choose 2 reds, which is half as many ways.

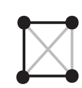
204. Let's find the three points of intersection. If we add our first equation to the third equation, we get $x + 2y + x - 2y = 8 + 0 \rightarrow 2x = 8 \rightarrow x = 4$. Substituting this for x in the first equation, we get $4 + 2y = 8 \rightarrow 2y = 4 \rightarrow y = 2$. So, one point of intersection is (4, 2). Subtracting the first equation from the second equation, we get $5x + 2y - (x + 2y) = 48 - 8 \rightarrow 4x = 40 \rightarrow x = 10$. Substituting this for x in the first equation gives us $10 + 2y = 8 \rightarrow 2y = -2 \rightarrow y = -1$ and a second intersection point of (10, −1). For the last point of intersection, adding the second equation and the third equation yields $5x + 2y + x - 2y = 48 + 0 \rightarrow 6x = 48 \rightarrow x = 8$. Substituting into the third equation, we see that $8 - 2y = 0 \rightarrow -2y = -8 \rightarrow y = 4$ and the third point of intersection is (8, 4). We need to determine which is the longest side. The distance between two points is determined by the formula $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$. But we can tell by looking at a plot of the points of intersection, without actually calculating the distances between them, that the greatest distance will be between the points (4, 2) and (10, − 1). The midpoint of the segment joining (4, 2) and (10, −1) is the point whose coordinates are the averages of the coordinates of the endpoints ((4 + 10)/2, (2 + (−1))/2) = (7, 1/2). Therefore, the sum of the coordinates of the midpoint of the longest side is 7 + 1/2 = **15/2.**

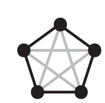
205. If there were eight girls and eight boys, there would be the same number of ways to choose three of each. If there are nine girls and seven boys there are ${}_9C_3$ = (9 × 8 × 7) ÷ (3 × 2 × 1) = 84 ways to choose three girls and ${}_7C_3$ = (7 × 6 × 5) ÷ (3 × 2 × 1) = 35 ways to choose three boys, but 84 ÷ 35 ≠ 6. With ten girls and six boys, then there are ${}_{10}C_3$ = (10 × 9 × 8) ÷ (3 × 2 × 1) = 120 ways to choose three girls and ${}_6C_3$ = (6 × 5 × 4) ÷ (3 × 2 × 1) = 20 ways to choose three boys. This works since 120 ÷ 20 = 6. The ratio of girls to boys, then, is 10/6 = **5/3**.

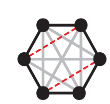
206. We will start with the prime factorization of 432, which is 3 × 3 × 3 × 2 × 2 × 2 × 2. We will use these factors to make four single-digit factors of 432, starting with the greatest possible four-digit number, which gives us 9 × 8 × 6 × 1 = 432. Unfortunately, 9 + 8 + 6 + 1 = 24, not 20. The next largest four-digit possibility gives us 9 × 8 × 3 × 2 = 432, but 9 + 8 + 3 + 2 = 22, not 20. Let's now try 9 × 6 × 4 × 2 = 432, which gives us 9 + 6 + 4 + 2 = 21, not 20. On the next try, we get 9 × 4 × 4 × 3 = 432 and 9 + 4 + 4 + 3 = 20. This means 9443 is the greatest number we can make. At the other extreme, we can make the number 2666, which meets the requirements that 2 × 6 × 6 × 6 = 432 and 2 + 6 + 6 + 6 = 20. The difference between these two numbers is 9443 − 2666 = **6777**.

207. The probability that Jebediah picks the fair coin is 1/2. The probability of flipping heads on a fair coin is 1/2. Therefore, the probability that he picks the fair coin and then flips three heads in a row is 1/2 × 1/2 × 1/2 × 1/2 = 1/16. The probability that Jebediah picks the two-headed coin is also 1/2. But the probability of flipping heads on the two-headed coin is 1. So the probability that he picks the two-headed coin and then flips three heads in a row is 1/2 × 1 × 1 × 1 = 1/2. The probability of flipping three heads in a row, then, is 1/16 + 1/2 = (1 + 8)/16 = 9/16. The probability this outcome occurs as a result of picking the fair coin is (1/16)/(9/16) = 1/16 × 16/9 = **1/9**.









208. One way to visualize this is using dots to represent people and colored lines between them to represent their relationship. A gray line is used to indicate that 2 people do not know each other, while a black line indicates that they do. We want to be able to color all of the lines connecting the dots without creating any monochromatic triangles (a connected group of three dots in which all three lines are the same color). Let's see if there is a scenario in which no 3 people all know each other and no 3 people are all strangers for 3, 4, 5, 6, … people in a room. The first three figures show an example of one such scenario involving 3, 4 and 5 people in a room. At 6 people it becomes difficult to find one such example. As the last figure shows, we can draw seven of the nine connections among 6 people without producing any monochromatic triangles. But either way we color the final two connections results in two monochromatic triangles. Let's verify that with 6 people there must be at least 3 people who all know each other or who all do not know each other. For a 6-person group every individual has a connection to 5 others. Of the 5 lines leaving each individual, at least 3 must be the same color. If we look at these 3 lines and 4 dots and draw in the other connections, we see that it is impossible to choose a color without creating a 3-person group that either are all connected by black lines or all connected by gray lines. Thus, the minimum number of people must be **6**. (Note: This is a special case of Ramsey's Theorem.)

209. We are given the formula $F = (9/5)C + 32$, but we need to convert Fahrenheit to Celsius, so $C = (5/9)(F - 32)$. Converting 80 °F to °C, we get (5/9)(80 − 32) = 240/9 °C. Converting 40 °F to °C, we get (5/9)(40 − 32) = 40/9 °C. To each of these numbers we need to add 273.15 to establish how far the temperature is from absolute zero. The percent decrease is (240/9 − 40/9)/(240/9 + 273.15) × 100% ≈ **7**%.

210. Three of the four faces of the tetrahedron are right triangles. Angles ADC, ADB and DBC are all 90-degree angles. The distance from B to the face ACD can be thought of as an altitude of triangle BDC drawn from B to side DC. The side lengths of triangle BDC are 12, 16, and 20 units, a multiple of the 3-4-5 Pythagorean Triple. The area of right-triangle BDC is 1/2 × 12 × 16 = 96 units^2. The length of side DC is 20 units, so for altitude h, we have $96 = 1/2 \times 20 \times h \rightarrow 96 = 10h \rightarrow h = 96/10 =$ **48/5** units.

Warm-Up 14

211. The number of chords that can be drawn in this circle with 8 points is ${}_8C_2$ = 8!/(6! × 2!) = (8 × 7)/(2 × 1) = **28** chords.

212. There are ${}_{10}C_3$ = 10!/(7! × 3!) = (10 × 9 × 8)/(3 × 2 × 1) = 120 ways to choose three cards from the deck of ten numbered cards. There are 8 sets of three consecutive numbers: 1-2-3, 2-3-4, 3-4-5, …, 7-8-9, 8-9-10. There are 9 sets of two consecutive numbers:1-2, 2-3, 3-4,…, 8-9, 9-10. Each of these pairs of consecutive numbers has 8 choices for the third number in the set, for a total of 9 × 8 = 72, but the sets of three consecutive numbers have been counted twice here. So, in all, there are 72 – 8 = 64 sets of three cards with two or more consecutive integers. Therefore, the probability of not drawing one of these sets is 1 – 64/120 = 56/120 = **7/15**.

213. It would take the original 12 people another 14 days to finish shearing the other 2/3 of the field of pine trees. The goal is to accomplish this in 6 days instead of 14 days, so 14/6 times as many people are needed. That would be 14/6 × 12 = 28 people, which means that 28 – 12 = **16** people must be added to the crew.

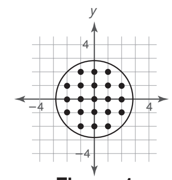


Figure 1

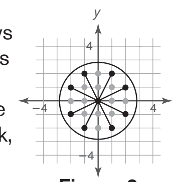
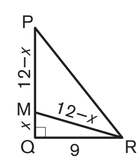


Figure 2

214. The general equation of a circle with center (0, 0) and radius r is $x^2 + y^2 = r^2$. So the circle's radius is $\sqrt{8} = 2\sqrt{2}$ units. This is the length of the diagonal of a square with side length 2 units, so our circle passes through the point (2, 2). Figure 1 shows the 21 lattice points that are in the interior of this circle. There are ${}_{21}C_3$ = 21!/(18! × 3!) = (21 × 20 × 19)/(3 × 2 × 1) = 7980/6 = 1330 ways to choose 3 points. That's the total number of outcomes possible. To determine the number of favorable outcomes, we will use complementary counting and find the sets of three points that *cannot* make a triangle. A triangle cannot be made if the three chosen points are collinear. Considering first the horizontal rows of points, there is ${}_3C_3$ = 1 way to choose all three points in both the top and bottom rows, and there are ${}_5C_3$ = 10 ways to choose three points from the three middle rows. That's 1 + 1 + 10 + 10 + 10 = 32 ways so far. By symmetry, there are also 32 ways to choose three collinear points from the vertical columns. Now let's consider the diagonals from the top left to the bottom right at a 45-degree angle. There are 1 + 4 + 1 + 4 + 1 = 11 ways to choose three collinear points on these diagonals. Using symmetry again, we can say there are another 11 ways on the other 45-degree diagonals. Besides these, there are another 4 sets of three collinear points that are shown in Figure 2. The total is 32 + 32 + 11 + 11 + 4 = 90 ways that do not work, so the other 1330 – 90 = 1240 sets of three points must make triangles. The probability, then, is 1240/1330 = **124/133**.

215. There is no four-digit number with all the digits the same for which the sum of the digits equals the product of the digits. So we are looking for a four-digit number with at least two distinct digits. Consider the four-digit number ABCD, where A > B ≥ C ≥ D and let's assume that A × B × C × D = A + B + C + D. Since B, C and D are all less than A, we can write the following inequality: A × B × C × D < A + A + A + A → A × B × C × D < 4 × A. Dividing both sides by A, we get B × C × D < 4. So, the product B × C × D can have a value of 1, 2 or 3. Given that B ≥ C ≥ D, the only possibilities for BCD, then, are 311, 211 and 111. Let's test these three options. If BCD = 311, we have A × 3 × 1 × 1 = A + 3 + 1 + 1 → 3 × A = A + 5 → 2 × A = 5 → A = 5/2, which cannot be true. If BCD = 211, we have A × 2 × 1 × 1 = A + 2 + 1 + 1 → 2 × A = A + 4 → A = 4, so the four-digit combination 4211 works. Finally, if BCD = 111, we have A × 1 × 1 × 1 = A + 1 + 1 + 1 → A = A + 3, which cannot be true. Therefore, the only combination of four digits that exists for which the sum of the digits equals the product of the digits is 4, 2, 1, 1. Since two of the four digits are the same, there are 4!/2! = (4 × 3 × 2 × 1)/ (2 × 1) = 24/2 = 12 possible arrangements of these four digits. Therefore, the number of four-digit numbers with this property is **12** numbers.

216. The three blue faces can intersect at one vertex, or they can be in a row of three adjacent faces of the cube. If the three blues intersect at a vertex, then the red, yellow and green sides can be placed in <u>2</u> different orientations that are mirror images of each other. If the three blues are in a row, then there are <u>3</u> ways the red, green and yellow faces can be arranged. That's a total of 2 + 3 = **5** cubes.

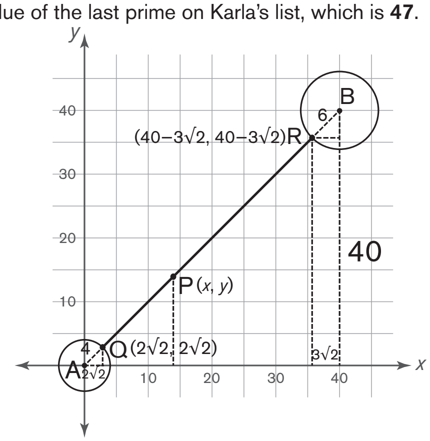


217. Let MQ = x . Since PQ = 12, it follows that MP = MR = 12 − x. Given that QR = 9, we can use the Pythagorean Theorem to find the side lengths of right triangle MQR. We have $x^2 + 9^2 = (12 - x)^2 \rightarrow x^2 + 81 = 144 - 24x + x^2 \rightarrow 81 = 144 - 24x \rightarrow 24x = 63 \rightarrow$ $x = 63/24 = 21/8$. Therefore, MQ = **21/8** units.

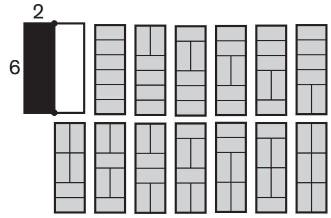
218. Karla could not have the number 2 on her list of six different primes since 2 would make the product of the first three primes even and the sum of the last three primes, which are all odd, must be odd. We are looking for a least possible value for the greatest number on Karla's list, so it makes sense to look at the least possible odd product of primes, which is 3 × 5 × 7 = 105. Now, let's look for three different primes with a sum of 105. Their average would be 105 ÷ 3 = 35. We should try to use two primes slightly above the average and one prime well below the average. After a bit of guessing and checking, we find that 17 + 41 + 47 = 105. When we consider the products 3 × 5 × 11 = 165 and 3 × 5 × 13 = 195, after more guessing and checking, we see that there aren't three primes between 11 and 47 whose sum is 165 or three primes between 13 and 47 whose sum is 195. So, we conclude that starting with the product 3 × 5 × 7 will yield the least possible value of the last prime on Karla's list, which is **47**.

219. The graph shows the initial positions of the two circles, with the segment from Q on circle A to R on circle B. Based on properties of 45-45-90 right triangles, Q has coordinates $(2\sqrt{2}, 2\sqrt{2})$ and R has coordinates $(40 - 3\sqrt{2}, 40 - 3\sqrt{2})$. Since the radius of circle B grows at twice the rate that the radius of circle A does, it follows that the distance along segment QR from Q to the point of tangency P is one-third the distance from Q to R. Similarly, for P(x, y), x must be

$$\frac{1}{3} \times \left(\left(40 - 3\sqrt{2}\right) - 2\sqrt{2}\right) + 2\sqrt{2} = \frac{40 - 5\sqrt{2}}{3} + 2\sqrt{2} = \frac{40 - 5\sqrt{2} + 6\sqrt{2}}{3} = \mathbf{\frac{40 + \sqrt{2}}{3}}.$$



220. Since none of the 1 × 2 rectangles can cross the center line, let's consider the number of arrangements of tiles on each side of the line. As the figure shows, there are 13 different ways to tile one side of the 6 × 4 rectangle. So, with 13 ways to tile the left side and 13 ways to tile the right side, there are 13 × 13 = **169** ways to tile the whole 6 × 4 rectangle.



Fractions Stretch

221. Dividing, we see that 25/100 = **1/4**.

222. Dividing, we see that 9/16 ÷ 3/8 = 9/16 × 8/3 = **3/2**.

223. Start by simplifying the expression under the radical symbol. We have $\sqrt{(3/11 \div 11/12)} = \sqrt{(3/11 \times 12/11)} = \sqrt{(36/11^2)}$ = **6/11**.

224. Of the 20 unit squares, 6 are shaded. That represents 6/20 = **3/10** of the squares.

225. The area of the triangle is $1/2 \times (3/2)w \times w = (3/4)w^2$. The area of the rectangle is $2w^2$. Thus, the triangle's area is (3/4)/2 = 3/4 × 1/2 = **3/8** of the rectangle's area.

226. The difference between 3/4 and 1/2 is 3/4 − 2/4 = 1/4, and 3/4 of that is 3/4 × 1/4 = 3/16. The common fraction that is 3/16 more than 1/2 is 8/16 + 3/16 = **11/16**.

227. The reciprocal of 1/(2 + 1/3) = 2 + 1/3 = 6/3 + 1/3 = **7/3**.

228. Let $x = 0.7\overline{5}$. Then $10x = 7.\overline{5}$ and $100x = 75.\overline{5}$. Subtracting, we see that $100x - 10x = 75.\overline{5} - 7.\overline{5} \rightarrow 90x = 68 \rightarrow x = 68/90 =$ **34/45**.

229. Let's start by simplifying the denominator. We get $\dfrac{1}{\frac{1}{n}+\frac{1}{3}} + \dfrac{1}{\frac{1}{3}+\frac{1}{n}} = \dfrac{1}{\frac{n+3}{3n}} + \dfrac{1}{\frac{n+3}{3n}} = \dfrac{2}{\frac{n+3}{3n}} = 2 \times \dfrac{3n}{n+3} = \dfrac{6n}{n+3}$. Since 1 divided by this fraction is simply the reciprocal $(n + 3)/(6n)$, we now have $(n + 3)/(6n) = 5/12$. Cross-multiplying gives us $12n + 36 = 30n \rightarrow 18n = 36 \rightarrow n =$ **2**.

230. Simplifying the left-hand side of the equation, we get $(2x - 2(x - 3))/(x - 3) = 4/(x + 2) \rightarrow 6/(x - 3) = 4/(x + 2)$. Cross-multiplying, we see that $6(x + 2) = 4(x - 3) \rightarrow 6x + 12 = 4x - 12 \rightarrow 2x = -24 \rightarrow x =$ **−12**.

Angles and Arcs Stretch

231. Vertex H is on the circle, so the measure of inscribed angle AHC is half the measure of intercepted $\overset{\frown}{AC}$. Since the regular nonagon divides the 360° of the circle into nine 40° arcs, it follows that $m\overset{\frown}{AC} = 80°$, making $m\angle AHC = 1/2 \times m\overset{\frown}{AC} = 1/2 \times 80 =$ **40°**.

232. Vertex B is on the circle, so the measure of inscribed ∠CBD is half the measure of intercepted minor $\overset{\frown}{BC}$. Since the measure of major $\overset{\frown}{BC}$ is 230°, it follows that minor $\overset{\frown}{BC}$ measures 360 − 230 = 130°, making $m\angle CBD = 1/2 \times m\overset{\frown}{BC} = 1/2 \times 130 =$ **65°**.

233. The measure of inscribed ∠ABE is 35°, so the measure of intercepted $\overset{\frown}{AE}$ is $2 \times m\angle ABE = 2 \times 35 = 70°$. Angle AXE has measure 15° and intercepts arcs AE and CD. So, $m\angle AXE = 1/2 \times (m\overset{\frown}{AE} - m\overset{\frown}{CD}) \rightarrow 15 = 1/2 \times (70 - m\overset{\frown}{CD}) \rightarrow 30 = 70 - m\overset{\frown}{CD} \rightarrow m\overset{\frown}{CD} = 70 - 30 =$ **40°**.

234. The measure of angle AXB, which intercepts major and minor $\overset{\frown}{AB}$, is 50°. If $m\overset{\frown}{AB}$ represents the measure of minor $\overset{\frown}{AB}$, then $m\angle AXB = 1/2 \times (m\overset{\frown}{AB} - (360 - m\overset{\frown}{AB})) \rightarrow 50 = 1/2 \times (2 \times m\overset{\frown}{AB} - 360) \rightarrow 100 = 2 \times m\overset{\frown}{AB} - 360 \rightarrow 2 \times m\overset{\frown}{AB} = 460 \rightarrow m\overset{\frown}{AB} = 460 \div 2 = 230$. Therefore, major $\overset{\frown}{AB}$ has measure **230°**.

235. Inscribed ∠ABD intercepts $\overset{\frown}{AD}$, which has measure 125°. Therefore, $m\angle ABD = 1/2 \times m\overset{\frown}{AD} = 1/2 \times 125 =$ **62.5°**.

236. Since $\overline{AC}$ is a diameter of circle O, we know that $m\overset{\frown}{AB} + m\overset{\frown}{BC} = 180°$. We are told that inscribed ∠BDC, which intercepts $\overset{\frown}{BC}$, has measure 40°. It follows, then, that $m\overset{\frown}{BC} = 2 \times 40 = 80°$, and $m\overset{\frown}{AB} = 180 - 80 =$ **100°**.

237. Inscribed ∠BAE intercepts $\overset{\frown}{ADB}$, so $m\angle BAE = 1/2 \times m\overset{\frown}{ADB}$. Arc ADB is composed of arcs ADC (a semicircle) and BC. From the previous problem, we know that $m\overset{\frown}{BC} = 80°$, so $m\overset{\frown}{ADB} = 180 + 80 = 260°$. It follows that $m\angle BAE = 1/2 \times 260 =$ **130°**.

238. Angles CFD and AFB intercept arcs CD and AB, respectively. These two angles are congruent and have measure equal to $1/2 \times (m\overset{\frown}{CD} + m\overset{\frown}{AB})$. Semicircle ADC is composed of arcs AD and CD of measure 125° and 180 − 125 = 55°, respectively. From Problem 236, we know that $m\overset{\frown}{AB} = 100°$. Therefore, $m\angle CFD = m\angle AFB = 1/2 \times (55 + 100) = 1/2 \times 155 =$ **77.5°**.

239. Angles CXD and AXB intercept arcs CD and AB, respectively. The measure of these two congruent angles is $1/2 \times (m\overset{\frown}{CD} + m\overset{\frown}{AB})$. We are told that $m\overset{\frown}{AB} = 110°$, and because AB = BD, it follows that $m\overset{\frown}{AB} = m\overset{\frown}{BCD} = 110°$. In addition, since $\overset{\frown}{BCD}$ is composed of arcs BC and CD, we have $m\overset{\frown}{BCD} = m\overset{\frown}{BC} + m\overset{\frown}{CD} \rightarrow 110 = 60 + m\overset{\frown}{CD} \rightarrow m\overset{\frown}{CD} = 110 - 60 = 50°$. Now that we have the measures of arcs AB and CD, we have $m\angle CXD = 1/2 \times (50 + 110) = 1/2 \times 160 =$ **80°**.

240. Angle DAC intercepts arcs CD and FG of circles O and P, respectively. So $m\overset{\frown}{FG} = m\overset{\frown}{CD} = 50°$. Semicircle AFG is composed of arcs AF and FG, so we have $m\overset{\frown}{AF} + 50 = 180 \rightarrow m\overset{\frown}{AF} = 180 - 50 = 130°$. Since ∠AOB, which intercepts arcs AF and FG, is in the exterior of circle P, it follows that $m\angle AOB = 1/2 \times (m\overset{\frown}{AF} - m\overset{\frown}{FG}) = 1/2 \times (130 - 50) = 1/2 \times 80 = 40°$. In addition, ∠AOB is a central angle of circle O that intercepts $\overset{\frown}{AB}$, so $m\overset{\frown}{AB} = 40°$ as well. Diameter AD of circle O creates a semicircle composed of arcs AB, BC and CD. Therefore, $m\overset{\frown}{AB} + m\overset{\frown}{BC} + m\overset{\frown}{CD} = 180 \rightarrow 40 + m\overset{\frown}{BC} + 50 = 180 \rightarrow 90 + m\overset{\frown}{BC} = 180 \rightarrow m\overset{\frown}{BC} = 180 - 90 =$ **90°**.

Bases Stretch

241. For base 9, the place values for a two-digit numeral are $9^1 = 9$ and $9^0 = 1$. Therefore, $24_9 = 2(9^1) + 4(9^0) = 2(9) + 4(1) = 18 + 4 =$ **22**.

242. For base 8, the place values for a two-digit numeral are $8^1 = 8$ and $8^0 = 1$. Therefore, $24_8 = 2(8^1) + 4(8^0) = 2(8) + 4(1) = 16 + 4 =$ **20**.

243. For base 7, the place values for a two-digit numeral are $7^1 = 7$ and $7^0 = 1$. Therefore, $24_7 = 2(7^1) + 4(7^0) = 2(7) + 4(1) = 14 + 4 =$ **18**.

244. The greatest power of nine that goes into 24 is 9, and $24 \div 9 = \underline{2}$ r $\underline{6}$. Therefore, $24 = 2(9^1) + 6(9^0) =$ **26** base 9.

245. The greatest power of eight that goes into 24 is 8, and $24 \div 8 = \underline{3}$ r $\underline{0}$. Therefore, $24 = 3(8^1) + 0(8^0) =$ **30** base 8.

246. The greatest power of seven that goes into 24 is 7, and $24 \div 7 = \underline{3}$ r $\underline{3}$. Therefore, $24 = 3(7^1) + 3(9^0) =$ **33** base 7.

247. For base 12, the place values for a four-digit numeral are $12^3 = 1728$, $12^2 = 144$, $12^1 = 12$ and $12^0 = 1$. The greatest power of 12 that goes into 4991 is 1728, and $4991 \div 1728 = \underline{2}$ r 1535. The greatest power of 12 that goes into 1535 is 144, and $1535 \div 144 = \underline{10}$ r 95. The greatest power of 12 that goes into 95 is 12, and $95 \div 12 = \underline{7}$ r $\underline{11}$. Recall, in base 12, that 10 = A and 11 = B. Therefore, 4991 = **2A7B** base 12.

248. In base 12, the first three place values are $12^2 = 144$, $12^1 = 12$ and $12^0 = 1$, and B = 11. So, $3BB_{12} = 3(144) + 11(12) + 11(1) = 432 + 132 + 11 = 575$ base 10. We now need to convert this to base 6. The greatest power of 6 that goes into 575 is $6^3 = 216$, and $575 \div 216 = \underline{2}$ r 143. The greatest power of 6 that goes into 143 is $6^2 = 36$, and $143 \div 36 = \underline{3}$ r 35. The greatest power of 6 that goes into 35 is 6, and $35 \div 6 = \underline{5}$ r $\underline{5}$. Therefore, $3BB_{12} =$ **2355** base 6.

249. If $523_b = 262$, then $5 < b < 10$ (since 262 is less than 523 and since 5 is a digit in base b). So the possible values for b are 6, 7, 8 and 9. If we try $b = 6$, we get $523_6 = 5(36) + 2(6) + 3(1) = 180 + 12 + 3 = 195$. If $b = 7$, we get $523_7 = 5(49) + 2(7) + 3(1) = 245 + 14 + 3 = 262$. Thus, $b =$ **7**. Alternatively, since $523_b = 5(b^2) + 2(b) + 3(1) = 5b^2 + 2b + 3$, we can solve the following quadratic equation: $5b^2 + 2b + 3 = 262 \rightarrow 5b^2 + 2b - 259 = 0 \rightarrow (5b + 37)(b - 7) = 0 \rightarrow 5b + 37 = 0$ or $b - 7 = 0 \rightarrow b = -37/5$ or $b = 7$. Since b must be positive and an integer, we conclude that the answer is $b =$ **7**.

250. Since 5 is a digit in base b, we know that $b > 5$, and we are told that $b < 10$. The possible values of b, then, are 6, 7, 8 and 9. We could try each of these to determine for which value of b both of the given equations hold true. Let's, instead, solve this problem algebraically. We know that $441_b = 4(b^2) + 4(b^1) + 1(b^0) = 4b^2 + 4b + 1$ and $351_b = 3(b^2) + 5(b^1) + 1(b^0) = 3b^2 + 5b + 1$, so we can write the following equations: $4b^2 + 4b + 1 = n^2$ and $3b^2 + 5b + 1 = (n - 2)^2$. Factoring the left-hand side of the first equation yields $(2b + 1)^2 = n^2$, so $n = 2b + 1$. Substituting this for n in the second equation, we get $3b^2 + 5b + 1 = (2b + 1 - 2)^2 \rightarrow 3b^2 + 5b + 1 = (2b - 1)^2 \rightarrow 3b^2 + 5b + 1 = 4b^2 - 4b + 1 \rightarrow b^2 - 9b = 0 \rightarrow b(b - 9) = 0 \rightarrow b = 0$ or $b - 9 = 0 \rightarrow b = 9$. Therefore, $b = 9$, and we have $441_9 = 4(81) + 4(9) + 1(1) = 324 + 36 + 1 = 361 = 19^2$, so $n = 19$. If we substitute this value for n in $(n - 2)^2$, we confirm that $(19 - 2)^2 = 17^2 = 289 = 3(81) + 5(9) + 1(1) = 351_9$. So $n =$ **19**.