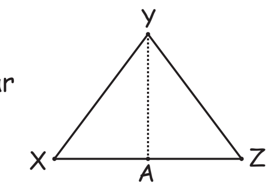
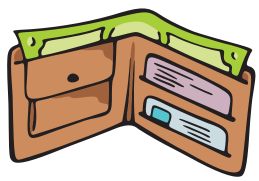
Warm-Up 1

1. $__________ If a total of twenty million dollars is to be divided evenly among four million people, how much money, in dollars, will each person receive?

2. __________ sq cm Triangle XYZ has side XZ = 6 cm. Segment YA is perpendicular to XZ. If AY is 4 cm, what is the area of triangle XYZ?



3. $__________



Wilhelmina went to the store to buy a few groceries. When she paid for the groceries with a $20 bill, she correctly received $4.63 back in change. How much did the groceries cost?

4. __________ multiples How many multiples of 8 are between 100 and 175?

5. __________ A three-digit integer is to be randomly created using each of the digits 2, 3 and 6 once. What is the probability that the number created is even? Express your answer as a common fraction.

6. __________ In the following arithmetic sequence, what is the value of m?

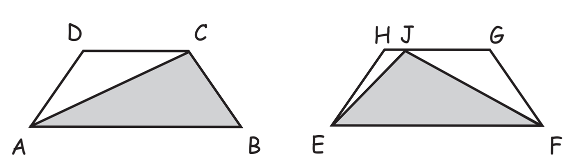
$-2, 4, m, 16, \ldots$

7. __________ A particular fraction is equivalent to $\frac{2}{3}$. The sum of its numerator and denominator is 105. What is the numerator of the fraction?

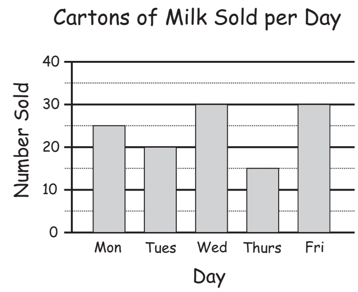
8. __________ What is the least natural number that has four distinct prime factors?

9. __________ areas The perimeter of a particular rectangle is 24 inches. If its length and width are positive integers, how many distinct areas could the rectangle have?

10. __________ sq units Trapezoid ABCD and trapezoid EFGH are congruent. What is the difference between the area of triangle ABC and the area of triangle EFJ?

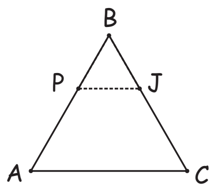


Warm-Up 2



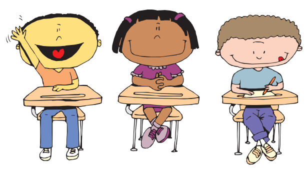
1. ________ cartons The bar graph shows the number of cartons of milk sold per day last week at Jones Junior High. What is the positive difference between the mean and median number of cartons of milk sold per day over the five-day period?

2. ________ inches



An equilateral triangle PBJ that measures 2 inches on each side is cut from a larger equilateral triangle ABC that measures 5 inches on each side. What is the perimeter of trapezoid PJCA?

3. ________ boys



Currently, $\frac{1}{4}$ of the members of a local club are boys, and there are 80 members. If no one withdraws from the club, what is the minimum number of boys that would need to join to make the club $\frac{1}{3}$ boys?

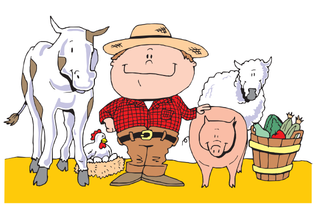
4. ________ outfits If William has 3 pairs of pants and 4 shirts, and an outfit consists of 1 pair of pants and 1 shirt, how many distinct outfits can William create?

5. ________ days Jessica reads 30 pages of her book on the first day. The next day, she reads another 36 pages. On the third day, she reads another 42 pages. If she continues to increase the number of pages she reads each day by 6, how many days will it take her to read a book that has 270 pages?

6. ________ A basket of fruit contains 4 oranges, 5 apples and 6 bananas. If you choose a piece of fruit at random from the basket, what is the probability that it will be a banana? Express your answer as a common fraction.

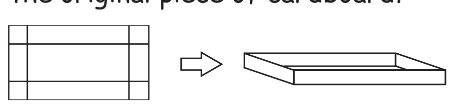
7. ________ What number is halfway between $\frac{5}{8}$ and $\frac{11}{16}$? Express your answer as a common fraction.

8. ________ pounds

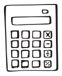


Farmer Fred read that his crop needs a fertilizer that is 8% nitrate for optimal yield. He needs to apply 4 pounds of nitrate per acre. If his field has 188 acres, how many pounds of fertilizer will Fred use?

9. ________ sq in Four squares are cut from the corners of a rectangular sheet of cardboard. It is then folded as shown to make a box that is 15 inches long, 8 inches wide and 2 inches tall. What was the area of the original piece of cardboard?



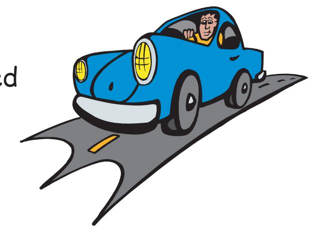
10. ________ degrees Two consecutive angles of a regular octagon are bisected. What is the degree measure of each of the acute angles formed by the intersection of the two angle bisectors?



Workout 1

1. $________ Sweaters cost $49 at a local department store, and the catalog department of the same store sells them for $44 online. The local department store charges 7% tax, and the catalog department charges $3.50 per sweater for shipping (but does not charge tax). How much would a person save by buying 6 sweaters through the catalog department rather than at the department store?

2. ________ mph If Jordan passed mile marker 138 at 5:08 pm and then passed mile marker 216 at 6:20 pm, what was her average speed in miles per hour?



3. ________ pounds A group of 5 women and 8 men weigh a total of 1921 pounds. If the women of the group have an average weight of 109 pounds, what is the average weight of the men in the group?

4. ________ womps On planet Wobble, 1 womp is equal in value to 3 wamps, and 2 wamps are equal in value to 5 wemps. How many womps are equal in value to 15 wemps?

5. ________ cubes How many positive three-digit perfect cubes are even?

6. ________ questions

Billy was playing a trivia game. According to the rules, Billy would receive 25 points for each question he answered correctly but would lose 50 points for each question he answered incorrectly. At the end of the game, Billy had a total of 450 points. He had 5 times as many questions correct as incorrect. If Billy answered every question, how many questions were asked in the game?

7. ________ What is the value of the following expression, expressed as a repeating decimal?
$\left(\frac{1}{9}+\frac{1}{7}\right)-\frac{2}{9}$

8. ________ Gary chooses a two-digit positive integer, adds it to 200, and squares the result. What is the largest number Gary can get?

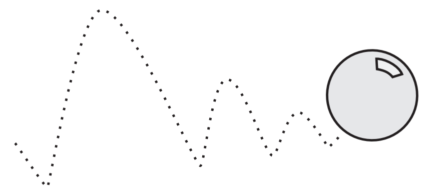
9. $________ The sale price of Kara's dress was $43.20 after a 40% discount. What was the original price of the dress?

10. ________ degrees The supplement of an angle is 5.5 times the complement of the angle. What is the measure of the angle?

Warm-Up 3

1. __________ Thirty percent of twice a number is 30. What was the original number?

2. __________ inches A bouncy ball bounces 16 feet high on its first bounce and exactly half as high on the next bounce. For each later bounce, the ball bounces half as high as on the previous bounce. How many inches high does the ball bounce on its seventh bounce?



3. $__________ Tamika's dad bought a Mustang convertible for $12,000 to celebrate his first job in 1978, which paid him $15,000 a year. If Tamika is to use the same fraction of her salary as her dad to buy a Mustang convertible, which now sells for $32,000, what will her first job have to pay?

4. __________ A rectangular prism has dimensions 4 by 6 by *x*. If the total surface area of the prism is 248 square units, what is the value of *x*?

5. __________ ways How many ways can the letters in FACTOR be rearranged so that the first and last letters are vowels?

6. __________ If $a \# b = (a + b)^2$, what is the value of $3 \# 1$?

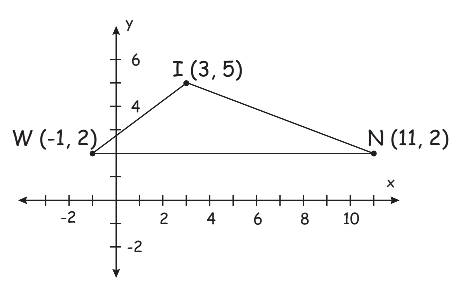
7. __________ The triangular array of positive integers shown continues indefinitely, with each row containing one entry more than the row above it. What is the sum of the two integers directly above 100?

4				
7	10			
13	16	19		
22	25	28	31	
34	37	40	43	46

8. __________ points In a class of 20 students, the average (arithmetic mean) score on a test is 84 points. If 6 students each scored 100 points and 4 students each scored 50 points, what is the average of the scores of the remaining students?

9. __________ If $\frac{a-2b}{3a-4b} = 5$, what is the ratio of a to b? Express your answer as a common fraction.

10. (___ , ___) If triangle WIN is reflected over the *x*-axis to create triangle W'I'N', what will be the coordinates of point I'? Express your answer as an ordered pair.



Warm-Up 4

Potato Chip Prices

Stem	Leaf
15	9
16	9
17	5 9
18	8 9
19	9 9 9 9 9
20	9
21	5 5 8 9
22	5 9 9
23	9
24	
25	9

Key
15 | 9 means $1.59

1. ________ bags A survey of the cost of one-pound bags of potato chips at five local supermarkets is shown in the stem-and-leaf plot. How many bags of chips cost less than the median price of a bag of chips?

2. ________ more rap songs Vivian has 45 songs on her MP3-player, 15 of which are alternative rock and the rest of which are rap. Malik has 60 songs on his MP3-player in the same ratio as Vivian's songs. How many more rap songs does Malik have on his MP3-player than Vivian has on hers?

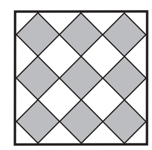
3. ________ pckgs Lashae will plant pansies and violets around the edge of her garden. She must use the same number of each plant, and she must use all of the flowers she buys. If pansies are sold in packages of 6 and violets are sold in packages of 10, what is the least number of packages of violets she could buy, given that she buys at least 1 package?

4. ________

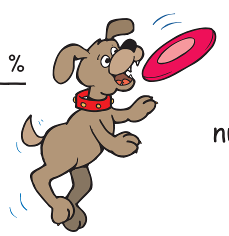
```
      1
     1 1
    1 2 1
   1 3 3 1
```

The first four rows of Pascal's triangle are shown to the left. If the first two numbers in a particular row of Pascal's triangle are 1 and 10, what is the fourth number of the next row?

5. ________ In the figure, the gray small squares and the white small squares are all the same size, and the outer corners of the outer gray squares lay on the edge of the largest square. What fraction of the largest square is shaded? Express your answer as a common fraction.



6. ________ %



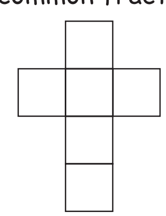
Last July there were 56 dogs for sale in the newspaper ads, and this July there were 84 dogs for sale. What is the percent increase in the number of dogs for sale?

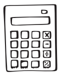
7. ________ What is the value of the expression $3x + 4y^3$ if $x = 3$ and $y = -2$?

8. ________ units Points A, B and C are collinear, with B between A and C. If the distance from point A to point B is 8 more than twice the distance from point B to point C, and the length of segment AC is 5 times the length of segment BC, how long is segment AC?

9. ________ The sum of two numbers is 3 times their positive difference. What is the ratio of the smaller number to the larger number? Express your answer as a common fraction.

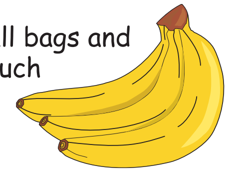
10. ________ units^3 The net shown is composed of 6 congruent squares and has a perimeter of 56 units. When the net is folded into a cube, what is the volume of the cube?





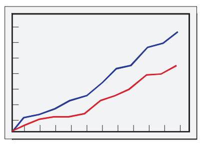
Workout 2

1. $ __________ Sara has 30 pounds of dried bananas that she wants to divide into small bags and sell. If each small bag holds $\frac{5}{8}$ of a pound and will sell for $1.58, how much will Sara bring in from sales if she sells all the bags?



2. __________ integers How many five-digit integers are perfect squares?

3. $ __________

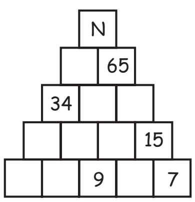


Four friends (Amanda, Barbara, Charles and Dylan) invested $3000, $5000, $7000 and $9000, respectively, in a certain start-up company. Profits will be divided in the same ratio as the investment. If the business made a profit of $72,000 this year, how much of the profit will the largest investor receive?

4. __________ points The mean score on Mr. Pascal's first semester exam for his 28 students is 73 points. If he removes the lowest score of 19 points, what is the adjusted class average (arithmetic mean)?

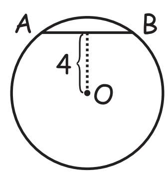
5. __________ The sum of four consecutive even integers is 596. What is the product of the least and greatest of these integers?

6. __________



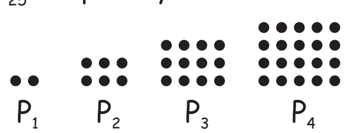
In this Number Wall, you add the numbers next to each other and write the sum in the block directly above the two numbers. What number will be in the block labeled N?

7. __________ sq units Chord AB of circle O is 6 units long and 4 units from center O. What is the area of circle O? Express your answer in terms of π.

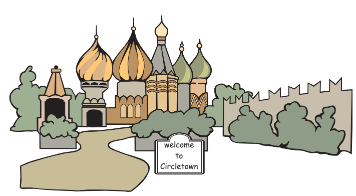


8. __________ Three fair six-sided number cubes, each with faces labeled 0, 2, 4, 6, 8, 10, are rolled. What is the probability that the sum of the numbers rolled is greater than 20? Express your answer as a common fraction.

9. __________ In the patterns of dots with P_1, P_2, P_3 and P_4, each successive pattern of dots has one more row and one more column. What is the ratio of the number of dots in P_{50} to the number of dots in P_{25}? Express your answer as a common fraction.



10. __________ people The population of Circletown has been growing at an annual rate of 5% for the last 5 years. The population is now 3105 people. What was the population of Circletown 5 years ago? Express your answer to the nearest whole number.

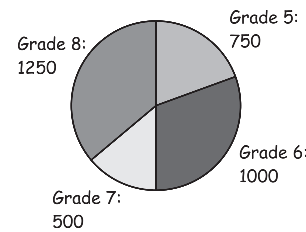


Warm-Up 5

1. __________ If Minh picks a letter of the alphabet at random, what is the probability it occurs in the name "Hattiesburg, Mississippi"? Express your answer as a common fraction.

2. __________ % The graph shows the amount of sales made by each grade at Memorial Middle School in the annual fundraiser. What percent of the total sales was made by the Grade 7 class? Express your answer to the nearest tenth.

Total Sales By Grade



3. __________ What is the integer value of $0 - 5 + 10 - 15 + 20 - 25 + 30 - \ldots + 240$?

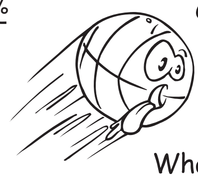
4. __________ In this additive magic square the sum of the three numbers in each row, in each column and along each diagonal is the same. What is the value of z?

		z
11	6	
		10

5. ______ sq units What is the area of the quadrilateral with vertices at (1, 1), (4, 1), (7, 5) and (4, 5)?

6. __________ If $a \# b = a^b - b^a$, what is the value of $3 \# 5$?

7. __________ %



Coach Kennedy had the starters of his basketball team practice free throws. Each player had to shoot 20 free throws. On Wednesday, 4 of the 5 starters did the free throw drill and averaged making 15 out of 20 attempted baskets. When the fifth starter did his drill, the team average dropped to 14 made out of 20 attempted. What percent of his free throws did the fifth starter make?

8. ______ ft per sec William is traveling at a rate of 75 miles per hour. How fast is William traveling in feet per second?

9. ______ pounds To create a more affordable line of her gourmet chocolates, Kindra mixes 10 pounds of chocolate worth \$1.50 per pound with another type of chocolate, worth \$9.00 per pound. She wishes to make a mixture of chocolate worth \$4.00 per pound. How many pounds of the \$9.00 chocolate must she add to create her new product?

10. ______ inches A solid is formed by placing a square pyramid with a base of 3 inches by 3 inches on top of a cube with edges of 3 inches. The volume of this combined solid is 54 cubic inches. What is the height of the pyramid?

Warm-Up 6

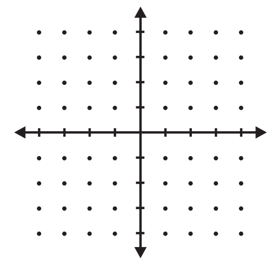
1. __________ What is the smallest positive integer that is divisible by 2, 5, 6 and 9?

2. __________ What is the value of $(0.07)^3$? Express your answer as a decimal to the nearest millionth.

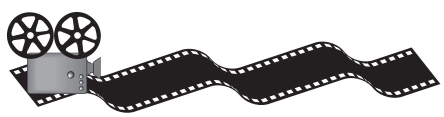
3. __________ If $x = \frac{a}{b}$ and $y = \frac{b}{a}$, then what is the square of the product of x and y?

4. __________ What is the only number that when added to its reciprocal is equal to 2?

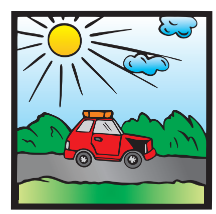
5. ______ points Two squares have centers at the origin of a coordinate plane and sides parallel to the axes. The smaller square has an area of 18 square units, and the larger square has an area of 50 square units. How many points with only integer coordinates are outside the smaller square region and inside the larger square region?



6. $ __________ Pedro earns $7.50 an hour at his job at Matrix Cinemas. If he works 20 hours this week and his employer withholds a total of 22% of his weekly pay for taxes and Social Security, what is his take-home pay for the week?



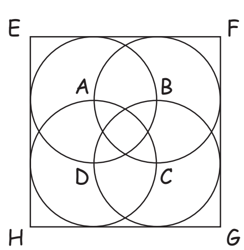
7. ________ pm Mary leaves New York City at 9:00 am, traveling to Charlotte, NC, at an average rate of 55 miles per hour. Simba leaves one hour later than Mary and follows Mary's route at an average rate of 65 miles per hour. At what time will Simba catch up to Mary?



8. (,) What ordered pair of positive integers (r, s) satisfies the equation $5r + 6s = 47$, such that $r > s$?

9. ______ index cards What is the minimum number of 3-inch by 5-inch index cards needed to completely cover a 3-foot by 4-foot rectangular desktop without cutting the index cards?

10. ________ sq ft



Points A, B, C and D are the centers of four circles, and they are also intersection points of these circles, as shown. Each circle has a radius of 6 feet and is tangent to two sides of square EFGH. What is the area of square EFGH?

Workout 3

1. __________ When the repeating decimal $0.\overline{38}$ is expressed as a common fraction, what is the sum of the numerator and denominator?

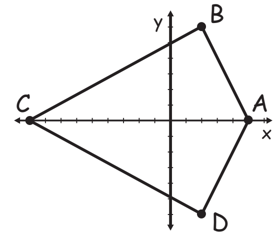
2. __________ The sum of three different two-digit prime numbers is 79. The largest of the numbers is 43. The difference of the other two is 10. What is the product of the three numbers?

3. __________ Five wooden disks are numbered -3, -2, 1, 4 and 7. If two disks are chosen at random, without replacement, what is the probability that their product is negative? Express your answer as a common fraction.

4. __________ The digits 1, 3, 5, 6, 7 and 9 are each used once and only once to form three two-digit prime numbers. What is the largest possible sum that can be formed?

5. __________ A cylinder and a rectangular prism have the same volume. If the length, width and height of the rectangular prism are doubled and the radius of the cylinder is doubled, what is the ratio of the volume of the new cylinder to the volume of the new rectangular prism? Express your answer as a common fraction.

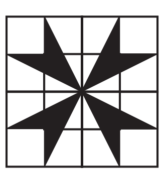
6. (,) A quadrilateral is formed by connecting the vertices A(5, 0), B(2, 6), C(–9, 0) and D(2, –6). What would be the coordinates of point C if you moved it to the right just enough to transform quadrilateral ABCD into rhombus ABCD? Express your answer as an ordered pair.



7. __________ am Three stoplights on different streets each operate on their own independent schedules, as follows: the first stoplight is red 1 minute out of every 2 minutes (1 minute red, then 1 minute green), the second is red 2 minutes out of every 3 minutes (2 minutes red, then 1 minute green) and the third is red 3 minutes out of every 5 minutes (3 minutes red, then 2 minutes green.) At 9:00 am each stoplight turns red. The lights are either red or green (don't worry about yellow). What time is it when the next 1-minute segment of time in which all three stoplights are red begins?

8. __________ If Sara makes 80% of the free throws she attempts, what is the probability that she misses exactly two of her next three free throws? Express your answer as a common fraction.

9. __________ sq units



A 4-unit by 4-unit grid has this pattern shaded in the design of a company's logo. What is the area of the shaded portion of the logo?

10. __________ What is the 2011th term of the arithmetic sequence –4, –1, 2, 5, . . . , where each term after the first is 3 more than the preceding term?

Systems of Equations Stretch

For problems 1-3, solve each of the following systems of equations. Describe the method you used. What other methods could you have used to solve each system? Express each answer as an ordered pair. Express any non-integer value as a common fraction.

1. $2x + 3y = 11$
   $3x - y = 11$

2. $\frac{2}{x} + \frac{3}{y} = 11$
   $\frac{3}{x} - \frac{1}{y} = 11$

3. $2x^2 + 3y^2 = 11$
   $3x^2 - y^2 = 11$

In problems 4 and 5, for what value of k does the linear system have no solution? Express any non-integer value as a common fraction.

4. $kx + 3y = 11$
   $3x - y = 11$

5. $2x + ky = 11$
   $3x - y = 11$

6. Given that $2a + b = 19$, $2c + d = 37$ and $b + d = 24$, what is the value of $a + b + c + d$?

7. Given that $a + b = 29$ and $ab = 204$, what is the value of $a^2 + b^2$?

8. Solve each of the following linear systems. Express your answer as an ordered pair.

a. $5x + 6y = 7$
   $8x + 9y = 10$

b. $x + 2y = 3$
   $4x + 5y = 6$

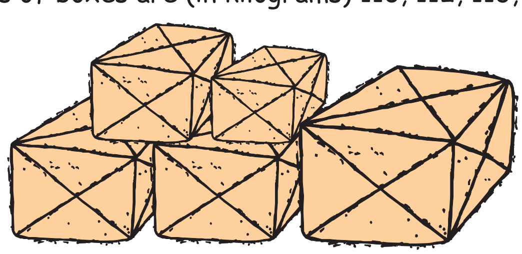
Many problems can be solved using a linear system of equations. For problems 9 and 10, use linear systems of equations to help you solve.

9a. For each equation below (i, ii, iii and iv), does there exist a solution (x, y) with positive integers x and y? In each case, either find all pairs of integers satisfying the equation or explain why none exist. Hint for the first equation: $x^2 - y^2 = (x + y)(x - y)$, so $x + y = 12$ and $x - y = 4$ is one case to check.

i. $x^2 - y^2 = 48$ ii. $x^2 - y^2 = 23$ iii. $x^2 - y^2 = 45$ iv. $x^2 - y^2 = 90$

b. In general, for what type of integers, n, does $x^2 - y^2 = n$ have at least one solution?

10. A shipping clerk wishes to determine the weights of each of five boxes. Each box weighs a different integer amount less than 100 kg. Unfortunately the only scales available measure weights in excess of 100 kg. The clerk therefore decides to weigh the boxes in pairs so that each box is weighed with every other box. The weights for the 10 pairs of boxes are (in kilograms) 110, 112, 113, 114, 115, 116, 117, 118, 120 and 121. From this information the clerk can determine the weight of each box. What are the weights of each of the five boxes?



What About Math?

In an effort to bridge the gap between the knowledge students gain from MATHCOUNTS and the important applications of this knowledge, MATHCOUNTS is providing a *What About Math?* section in Volume I of the *MATHCOUNTS School Handbook*. This year the problems are from the fields of **engineering** and **statistics**. Written by actual professionals in these fields, these problems illustrate how students can put the skills they are learning today into practice in the near future.

This year's problems and solutions were written for MATHCOUNTS by members of the Association for Unmanned Vehicle Systems International (AUVSI), The Actuarial Foundation (TAF) and the Consumer Electronics Association (CEA).

AUVSI Foundation Writers/Editors: *Wendy Amai, Sandia National Laboratories; Dr. Kenneth Berry, Assistant Director of the Science and Engineering Education Center, University of Texas at Dallas; Angela Carr, Web Services Manager, AUVSI; Dr. Deborah A. Furey, DARPA Program Manager; Lisa Marron, Sandia National Laboratories and Dr. Dave Novick, Sandia National Laboratories*

TAF Writers/Editors: *Charles Cicci, ACAS, MAAA; Kevin Cormier, FCAS, MAAA; Jeremy Fogg; Mubeen Hussain; Jeremy Scharnick, FCAS, MAAA; John Stokesbury, FSA, FCA, EA, MAAA; Pete Rossi, FSA, FCA, CERA, MAAA and Patricia Teufel, FCAS, MAAA.*

CEA Writers/Editors: *Bill Belt; Deepak Joseph; Brian Markwalter, PE and Dave Wilson*

ASSOCIATION FOR UNMANNED VEHICLE SYSTEMS INTERNATIONAL FOUNDATION (WWW.AUVSIFOUNDATION.ORG)

The AUVSI Foundation is a nonprofit, charitable organization that was established to support the educational initiatives of the Association for Unmanned Vehicle Systems International (AUVSI). The AUVSI Foundation's primary objective is to develop educational programs that attract and equip students for careers in robotics. To do this, the AUVSI Foundation provides K-12 and university level students with hands-on educational activities that highlight the world of robotics while emphasizing STEM curriculum–science, technology, engineering and math.

Currently, the AUVSI Foundation directly sponsors five annual advanced robotics competitions that attract nearly 200 student teams, representing more than 2000 students from schools around the world. Last year, nearly $100,000 in prize money was awarded through these student competitions. For more information on the AUVSI Foundation and its educational programs, please visit www.auvsifoundation.org.

THE ACTUARIAL FOUNDATION (WWW.ACTUARIALFOUNDATION.ORG)

The Actuarial Foundation (TAF) was founded in 1994 as the charitable arm of the actuarial profession. TAF's mission is to develop, fund and execute education and research programs that serve the public by harnessing the talents of actuaries. Through the Foundation's youth education math activities, actuaries assist with the development of math materials that help students develop a love for math and embrace mathematical concepts in a fun and engaging way in order for them to understand how math is used in the "real world." If you like math – consider becoming an actuary. "Actuary" was rated the number two profession in the most recent *Jobs Rated Almanac*. The actuarial profession's ability to continue providing competent service in the future requires supporting mathematics achievement among today's students. The Actuarial Foundation is proud to support and sponsor MATHCOUNTS!

CONSUMER ELECTRONICS ASSOCIATION (WWW.CE.ORG)

The Consumer Electronics Association (CEA) is the preeminent trade association promoting growth in the $165 billion U.S. consumer electronics industry. More than 2000 companies enjoy the benefits of CEA membership, including legislative advocacy, market research, technical training and education, industry promotion, standards development and the fostering of business and strategic relationships. CEA also sponsors and manages the International CES–The Global Stage for Innovation. All profits from CES are reinvested into CEA's industry services.

What About Math?

AUVSI Foundation

1a. ________kg Robots often have two treads (like tank treads) instead of four wheels. You can calculate the amount of weight a tread or wheel is carrying per square inch of tread or wheel by dividing the weight of the robot by the total area of the tread or wheel on the ground.

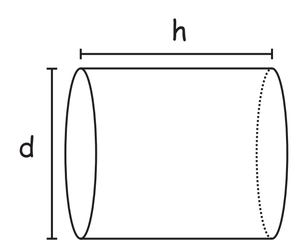
If a robot weighs 120 kg and each of the robot's 2 treads touches an area of ground that is 30 cm long and 5 cm wide, how much weight is carried over each square centimeter of tread? Express your answer as a decimal to the nearest tenth.

1b. ________kg Compare this to the same robot with 4 wheels (instead of tread) that each touch a section of ground that is 3 cm × 5 cm. How much weight is on each square centimeter of the wheels?

1c. ________cm A particular robot wheel is 6 cm in diameter. The motor attached to the wheel runs at a speed of 120 rpm (rotations per minute). How many centimeters will the robot travel in 30 seconds? Express your answer as a decimal to the nearest tenth.

2. _________kg The buoyancy of an object is defined as the mass of the displaced fluid. A particular underwater robot that has a mass of 10 kg has a volume equivalent to the right circular cylinder shown. The density of water is 1000 kg/m^3, and the volume of a right circular cylinder is defined as $V = \pi r^2 h$.

For d = 9 cm and h = 100 cm (use 3.14 for π), determine if this robot will sink or float. How much flotation will need to be added (or mass will need to be added/removed) to get this robot to be neutrally buoyant in water? Express your answer as a decimal to the nearest ten thousandth. Note: For an object to be neutrally buoyant, the mass of the object must be equal to the mass of an identical volume of water.



3a. ______miles An unmanned aerial vehicle (UAV) carries 45 gallons of fuel. It burns fuel at a rate of 3 gph (gallons/hour) and flies at a speed of 160 mph (miles/hour). We want to fly it to a location X, fly it in the area for 1 hour, and then fly it back to its launching point.

What is the greatest number of miles away from the UAV's launching point can location X be?

3b. ______miles How far away can location X be if we want the UAV to arrive back home with a fuel reserve of 0.5 hour?

4a. ______ frames per sec A communication network can carry 128,000,000 bits per second. A digital video camera on the network makes pictures that are 640 pixels wide by 480 pixels high. Each pixel is composed of 3 bytes. A byte is 8 bits. A picture is called a "frame."

How many frames per second from the camera can this network carry? Express your answer to the nearest whole number.

4b. ______ pixels We can increase the frame rate if we decrease the size of the picture. If we wanted to send back square frames at a rate of 30 frames/second, how many pixels wide would a square frame (same size high as wide) be? Express your answer as a decimal to the nearest tenth.

5a. ______ meters Obstacles such as fallen trees, fences and ditches can be difficult for wheeled robotic vehicles to overcome. Imagine that a specially equipped robot could jump over such things. At the peak of the jump, the robot's vertical velocity is 0 m/s. Standard gravity on Earth, g, is 9.8 m/s^2.

If the time it takes to fall from the peak is 1.7 s, how high did the robot jump? Height is equal to $\frac{1}{2}(g)(t)^2$, where g = gravity and t = time to fall to the ground, in seconds. Express your answer as a decimal to the nearest hundredth.

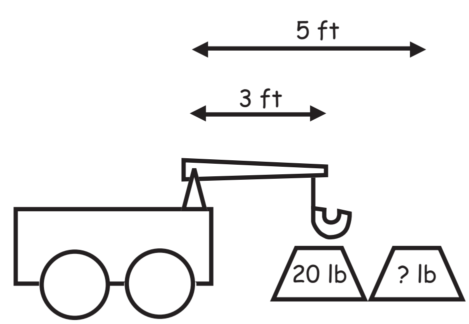
5b. ______ meters per second How fast, in meters per second, is the robot traveling when it hits the ground? Velocity = $(g)(t)$. Express your answer as a decimal to the nearest hundredth.

6. ______ min ______ sec In 2003, NASA launched the rover Spirit onto the surface of Mars to explore our planetary neighbor. Near the end of January 2010, Mars was 99.33 million km away from Earth. Assume the speed of communication is at the speed of light, which is 300,000,000 m/s.

If a researcher on Earth requested information from Spirit about its temperature, how long did the researcher have to wait for a reply after sending the request? Express your answer in the form x minutes y seconds, where x and y are whole numbers.

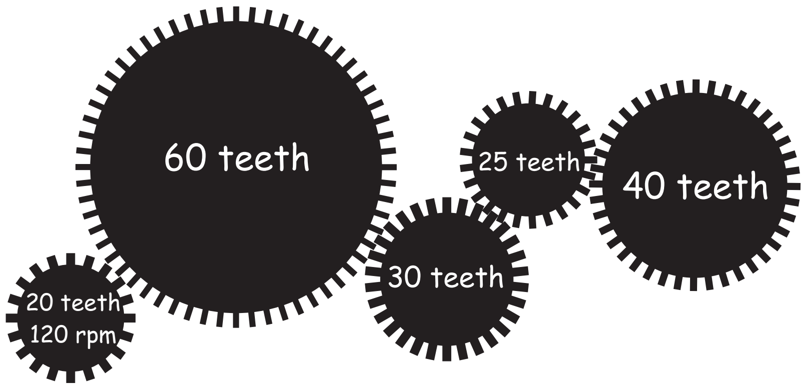
7. ______ feet A robot has a battery life of 3 hours while carrying 0 pounds. The robot's maximum velocity is 20 feet per minute. For every pound that the robot needs to carry, the total amount of battery time the robot will operate while traveling at maximum velocity decreases by 8 minutes. Starting with a fully charged battery, how many feet can the robot travel at maximum velocity while carrying 10 pounds?

8. ______ pounds A small ground robot has a telescoping lifting arm with a motor that can pick up a 20-pound weight when it is 3 feet away. The motor can support a maximum moment, or torque, of 60 foot-pounds (moment = weight × distance). How much weight, in pounds, could the arm lift at 5 feet away?



9. ______ foot-pounds Torque is a measure of "rotational force," and it is equal to force times distance. A motor at the end of an 8-inch robot arm is lifting a 3-pound weight. How much torque is the motor generating? Note: Torque is measured in foot-pounds.

10a. ________ Gears perform two important duties. First, they can increase or decrease the speed of a motor. Second, they can decrease or increase the power transferred. Gears with the same size teeth are usually described not by their physical size but by the number of teeth around the circumference. The gear ratio is the relationship between the numbers of teeth on two gears that are meshed. Gear ratio = (number of teeth on output)/(number of teeth on input). When multiple gears are involved, the gear ratios from each pair of meshed gears are multiplied to calculate the gear train's gear ratio.



What is the gear ratio of the gear train shown if the rightmost gear is the input? Express your answer as a common fraction.

10b. ______ rpm The speed of an output gear is equal to (input speed) × (1 ÷ gear ratio). If a motor with 120 rpm (revolutions per minute) is attached to the leftmost gear (input), what is the speed of the final rightmost gear that is the output?

What About Math?

The Actuarial Foundation

1. ________ The probability that a particular state will have neither a hailstorm nor a tornado in a given month is 55%. In the same period, the probability of a hailstorm is 35% and the probability of a tornado is 25%. If the probability of a hailstorm and the probablity of a tornado are independent but *not mutually exclusive*, what is the probability of both a hailstorm and a tornado occurring in a given month? Express your answer as a decimal to the nearest hundredth.

Use the following information for problems 2 and 3.

The amount of damage (in millions of dollars) that could be caused by an earthquake in the fictitious country of Shakalot is shown in the table below.

Dollar Value of Damage (millions)	**Probability**
5	10%
10	15%
25	20%
50	30%
75	20%
100	4%
150	1%

2. $______ million What is the expected (mean) damage amount, in millions of dollars, for an earthquake in Shakalot? Express your answer as a decimal to the nearest tenth.

3. $______ million What is the interquartile range, in millions of dollars, for this data?

4. ______ policy-holders An insurance company has sold 10,000 policies. The policyholders are classified using gender (Male or Female) and age (Young or Old). Of these policyholders, 3000 are Old, 4000 are Young Males and 4000 are Female. How many of the company's policyholders are Old Females?

5. $________ Sneaky Joe has a loaded die with the numbers 1, 2 and 3 each having a probability of 1/4 of being rolled and the numbers 4, 5 and 6 each having a probability of 1/12 of being rolled. No one but Joe knows about the loaded die.

Sneaky Joe offers to take the following bet with you: You roll the die. If the result is an even number, you win $$x$; if it's an odd number, you lose and pay Joe $5.

How much should Sneaky Joe pay you for a win to make this a fair bet?

6. ________% A certain disease is expected to infect 1 out of every 10,000 individuals in a country. A test for the disease is 99.5% accurate. It never gives a false indication when it is negative, so 0.5% of the people who take the test will get inaccurate readings, all of which will be false positives (meaning that the people test positive but do not have the disease).

Let us suppose you test positive; what is the probability that you actually have the disease? Express your answer as a percent to the nearest whole number.

Use the following information for problems 7–9.

Frequency = the ratio of the number of claims to the number of policies
Severity = the mean size of a group of claims
Pure Premium = Frequency × Severity
Charged Premium = Pure Premium + Expenses + Profit

Suppose there are 50 policies that generate the following five claims in 2009:

$7500
$5000
$3000
$11,500
$8000

7. ________ claims/policies What is the frequency? Express your answer as a decimal to the nearest hundredth.

8. $________ What is the severity of the five claims?

9. $________ How much premium should be charged if 15% of the charged premium is to be used for expenses and 5% of the charged premium is for profit?

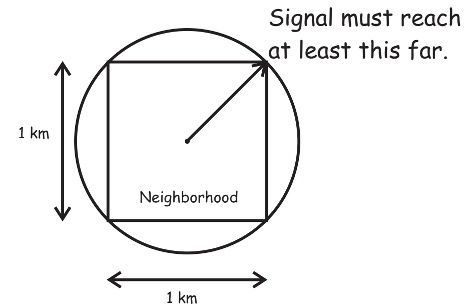
10. ________ days AutoMakers, Inc. is a car manufacturer and currently has three operational plants cleverly named Plant A, Plant B and Plant C. Plant A can produce 100 cars a day. Plant B can produce 80 cars a day. Plant C can produce only 70 cars a day.

If all three plants are running at full production capacity, how many full days are needed to produce 1600 cars?

What About Math?

Consumer Electronics Association

A provider of wireless Internet service will offer service to a neighborhood that is shaped like a square and is 1 km (1000 m) long on each side. The service provider will be using a tower that is located precisely in the middle of the square. The signal from the tower will travel an equal distance in all directions, producing a circular coverage area.



To function correctly, a consumer's wireless Internet device must receive a signal from the wireless Internet service provider that is at least 1 microwatt (1×10^{-6} watt) in strength. Radio frequency signals lose their strength as they travel over a distance according to the following formula:

Received signal strength in watts = Transmitted signal strength in watts ÷ $[4\pi d/\lambda]^2$, where λ is the wavelength of the wireless Internet signal in meters and d is the distance from the transmitter to the receiver in meters.

The wavelength (λ) of a radio frequency signal in meters is equivalent to the speed of light in meters per second divided by the frequency of the signal in hertz (Hz).

The speed of light is approximately 3×10^8 meters per second.

The wireless Internet service provider will be using the frequency 900 megahertz (MHz), which is equivalent to 900,000,000 Hz.

1. ______ watts How strong does the signal sent by the transmitter have to be in order to provide at least one microwatt to every receiver within the 1-square-kilometer neighborhood? Express your answer to the nearest whole number. Note: Disregard the height of the transmitters.

An electric circuit in a house goes from the circuit breaker box in the basement to the kitchen. In the kitchen there are four outlets connected to this circuit. These are the only things in this circuit.

The wire in the wall from the basement to the kitchen is able to handle 15 amps of current without becoming too hot. The circuit breaker for this circuit in the basement protects the wiring in the wall from damage. The ciruit breaker will trip if more than 15 amps of current flows through the circuit.

The house has standard 120-volt electric service. In the kitchen there is one thing connected to the outlets in the circuit—an 1100-watt toaster. The power required for operating the toaster will be pulled from the total power available to the circuit. The homeowner wants to purchase a coffeemaker and connect it to another outlet in the same circuit.

2. ______ watts What is the maximum power, in watts, that the coffeemaker can consume without causing the circuit breaker in the basement to trip when both the coffeemaker and the toaster are operating simultaneously? Note: Power = voltage x current (that is, watts = volts × amperes).

Plug-in hybrid electric vehicles (PHEVs) combine one or more electric motors and a gas or diesel engine for propulsion. Energy is stored in a battery pack for the electric motor and in the gas tank for the combustion engine. The combination allows short trips to be made entirely on electric power and raises the effective fuel efficiency of the car. The battery in a typical PHEV stores 9000 watt-hours (W-h) of energy.

Some additional information you will need:

A house has standard 120-volt electric service.
power = voltage × current (that is, watts = volts × amperes)
energy = power × time (that is, watt-hours = watts × hours)

A standard wall outlet with a 15-ampere circuit breaker can supply 12 amperes for continuous charging, which avoids tripping the circuit breaker (circuit breakers are set 25% above the maximum continuous current a circuit is designed to carry). Electric vehicle engineers call charging from a standard outlet Level 1 charging.

3. ________ hours How long will it take to charge a PHEV with a 9000-Wh-battery from a standard wall outlet in your home? Express your answer as a decimal to the nearest hundredth.

Level 2 charging uses a 240-volt outlet that typically supplies 32 amperes of current.

4. ________ hours How long will it take to charge the same 9000-Wh-PHEV (discussed in question 3) with a Level 2 charging outlet? Express your answer as a decimal to the nearest hundredth.

Televisions use electric power while turned off and while turned on. The small amount of power used when a TV is off allows the TV to respond to a command from a remote control. Most of the power used when the TV is turned on goes toward producing the light for the picture we see on the screen. The average TV spends 5 hours a day turned on and 19 hours a day turned off. A 55-inch TV uses 197 watts/hour when it is on and 0.4 watt/hour when it is off. Thus, a 55-inch TV uses 992.6 watts per day, which can be calculated by adding together the power used while the TV is on and the power used while the TV is off each day. The power used while the TV is on is found by multiplying the number of hours the TV is on times the power used per hour while it is on. A corresponding formula applies for the power used while the TV is off.

5. ________ watts How many watts per day does a small TV use if it uses 33 watts/hour while on and 0.4 watt/hour while off? Assume that the TV is on for 5 hours and off for 19 hours. Express your answer as a decimal to the nearest tenth.

Digital displays (such as TVs and computer monitors) are built according to standard dimensions (width and height) and resolutions (pixels). The width and height dimensions of a display can also be expressed as a ratio called the aspect ratio. The aspect ratio of a display is the ratio of the width of the image to its height, expressed as two numbers separated by a colon. The resolution of a display is the number of distinct pixels (smallest elements of a screen) that can be displayed in each dimension.

For example, a display that has resolution of 800×600 pixels, or 800 pixels in the width dimension and 600 pixels in the height dimension has an aspect ratio of 4:3. This is because the aspect ratio 800:600 can be reduced to 4:3.

6. __________ If a high-definition digital TV has a resolution of 1920×1080 pixels, what is its aspect ratio? Express your answer in the form a:b, where a and b are positive integers and have no common factors other than 1.

Internet speed to a home is 25 megabits per second (Mbps) peak throughput. Viewing or using certain common websites takes up varying bandwidth as shown in the list below:

(a) Viewing MySpace takes up 1 Mbps
(b) Using Pandora takes up 5 Mbps
(c) Viewing Facebook takes up 2 Mbps
(d) Viewing Amazon On Demand takes up 24 Mbps
(e) Viewing YouTube-HD takes up 12 Mbps

For example: Viewing Myspace (1 Mbps) and listening to Pandora (5 Mbps) at the same time uses 1 Mbps + 5 Mbps = 6 Mbps of bandwidth of the 25 Mbps available. This is because bandwidth usage is additive. If a group of websites exeeds the available bandwidth, they cannot all be used simultaneously.

7. ______ websites What is the maximum number of distinct websites from the above list that can be simultaneously used at the home?

ANSWERS TO HANDBOOK AND WHAT ABOUT MATH? PROBLEMS

In addition to the answer, we have provided a difficulty rating for each problem. Our scale is 1-7, with 7 being the most difficult. These are only approximations, and how difficult a problem is for a particular student will vary. Below is a general guide to the ratings:

Difficulty 1/2/3 - One concept; one- to two-step solution; appropriate for students just starting the middle-school curriculum.
4/5 - One or two concepts; multistep solution; knowledge of some middle-school topics is necessary.
6/7 - Multiple and/or advanced concepts; multistep solution; knowledge of advanced middle-school topics and/or problem-solving strategies is necessary.

Warm-Up 1

Answer	Difficulty	Answer	Difficulty	Answer	Difficulty
1. 5 or 5.00	(1)	4. 9	(1)	7. 42	(2)
2. 12	(1)	5. $\frac{2}{3}$	(1)	8. 210	(1)
3. 15.37	(1)	6. 10	(1)	9. 6	(3)
				10. 0	(2)

Warm-Up 2

Answer	Difficulty	Answer	Difficulty	Answer	Difficulty
1. 1*	(1)	4. 12	(1)	7. $\frac{21}{32}$	(2)
2. 13	(1)	5. 6	(1)	8. 9400	(2)
3. 10	(2)	6. $\frac{2}{5}$	(1)	9. 228	(2)
				10. 45	(3)

Workout 1

Answer	Difficulty	Answer	Difficulty	Answer	Difficulty
1. 29.58	(2)	4. 2	(2)	7. $0.\overline{031746}$	(2)
2. 65	(3)	5. 2	(2)	8. 89,401	(1)
3. 172	(2)	6. 36	(3)	9. 72 or 72.00	(3)
				10. 70	(4)

* The plural form of the units is always provided in the answer blank, even if the answer appears to require the singular form of the units.

Warm-Up 3

Answer	Difficulty				
1. 50	(2)	4. 10	(3)	7. 140	(2)
2. 3	(2)	5. 48	(3)	8. 88	(2)
3. 40,000 or 40,000.00	(2)	6. 16	(2)	9. $\frac{9}{7}$	(4)
				10. $(3, -5)$	(2)

Warm-Up 4

Answer	Difficulty				
1. 6	(1)	4. 165	(4)	7. -23	(4)
2. 10	(2)	5. $\frac{1}{2}$	(3)	8. 20	(3)
3. 3	(1)	6. 50	(2)	9. $\frac{1}{2}$	(3)
				10. 64	(2)

Workout 2

Answer	Difficulty				
1. 75.84	(2)	4. 75	(2)	7. 25π	(4)
2. 217	(3)	5. 22,192	(2)	8. $\frac{35}{216}$	(4)
3. 27,000 or 27,000.00	(2)	6. 132	(2)	9. $\frac{51}{13}$	(3)
				10. 2433	(3)

Warm-Up 5

Answer	Difficulty
1. $\frac{6}{13}$	(1)
2. 14.3	(2)
3. 120	(3)
4. 7	(3)
5. 12	(3)
6. 118	(2)
7. 50	(3)
8. 110	(3)
9. 5	(4)
10. 9	(4)

Warm-Up 6

Answer	Difficulty
1. 90	(2)
2. 0.000343	(2)
3. 1	(2)
4. 1	(3)
5. 24	(3)
6. 117 or 117.00	(2)
7. 3:30	(3)
8. (7, 2)	(4)
9. 116	(3)
10. 324	(3)

Workout 3

Answer	Difficulty
1. 137	(3)
2. 12,857	(2)
3. $\frac{3}{5}$	(3)
4. 211	(3)
5. $\frac{1}{2}$	(4)
6. (-1, 0)	(3)
7. 9:06	(3)
8. $\frac{12}{125}$	(5)
9. 4	(3)
10. 6026	(3)

System of Equations Stretch

Answer

1. (4, 1)
2. ($\frac{1}{4}$, 1)
3. (±2, ±1)
4. -9

5. $-\frac{2}{3}$
6. 40
7. 433

8a. (-1, 2)
8b. (-1, 2)

9a. i. (13, 11), (8, 4), (7, 1)
ii. (12, 11)
iii. (23, 22), (9, 6), (7, 2)
iv. no solution

9b. *n* is odd or a multiple of 4.

10. 54, 56, 58, 59, 62

What About Math? - AUVSI Foundation

Answer

1a. 0.4; b. 2; c. 1131.0

2. 3.6415

3a. 1120; b. 1080

4a. 17; b. 421.6

5a. 14.16; b. 16.66

6. 11 min 2 sec
7. 2000
8. 12
9. 2

10a. $\frac{1}{2}$; b. 60

What About Math? - TAF

Answer

1. 0.15
2. 42.5
3. 40
4. 1000
5. 7 or 7.00
6. 2
7. 0.10
8. 7000 or 7000.00
9. 875 or 875.00
10. 7

What About Math? - CEA

Answer

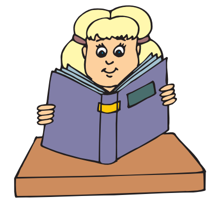
1. 711
2. 700
3. 6.25
4. 1.17
5. 172.6
6. 16:9
7. 4

Warm-Up 7

1. __________ Melinda chooses a three-digit positive integer, subtracts it from 3000, and triples the result. What is the largest integer Melinda can get?

2. __________ Each term of a sequence, after the first term, is three less than the square of the preceding term. If the first term of the sequence is 2, what is the 2011th term?

3. ______ hours Susan reads at a rate of 240 words per minute. How many hours will it take her to read a 480-page book that averages 600 words per page?



4. __________ A fair coin is to be flipped 5 times. What is the probability that the result will not be 5 heads in a row? Express your answer as a common fraction.

5. ______ degrees What is the degree measure of the only angle that is congruent to its complement?

6. __________ If $a + 2b = 11$ and $a - b = -4$, what is the value of $4a - b$?

7. __________$_2$ What is the value of 1011_3 expressed as a numeral in base 2?

8. ______ integers From the set of digits {1, 2, 3, 4, 5} three-digit, positive integers are formed, none of which has repeating digits. How many such odd integers can be made?

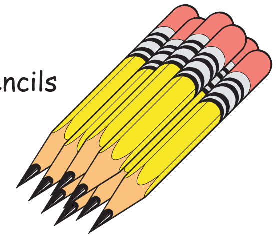
9. _________% Triangle *ABC* has vertices with coordinates (1, 0), (5, 0), and (3, 7). Kadim accidentally switched the *x*- and *y*-coordinates of every single vertex when calculating the area of the triangle. What is the percent of change between the area of the original triangle *ABC* and the area of the new triangle Kadim made?

10. ______ times

Eight students met for the first time at a MATHCOUNTS competition. Every one of them exchanged cell phone numbers with each of the other seven students, and they all entered the phone numbers into their phones. How many times were cell phone numbers entered into phones among the group of 8 students?

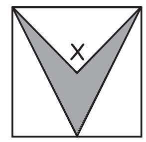
Warm-Up 8

1. \$__________ At the school store pencils and pens have different prices. Six pencils and four pens cost \$4.30. However, four pencils and six pens cost \$5.20. What is the cost of five pencils and five pens?



2. __________ What term is the smallest five-digit palindrome in the arithmetic sequence 2, 7, 12, 17, … ?

3. __________ sq yds A chevron is inscribed in a square of side 12 yards where point X is at the center of the square and the upper corners of the chevron touch the upper corners of the square. What is the area of the chevron?



4. __________ On a number line, what is the positive difference between the two numbers that are the trisection points of the line segment with endpoints at $\frac{1}{8}$ and $\frac{3}{4}$? Express your answer as a common fraction.

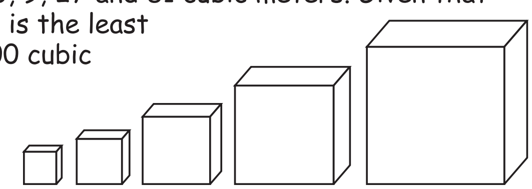
5. __________ minutes Manny can mow his one-acre yard in 1.5 hours on his riding lawn mower. Timmy takes five times as long to mow Manny's yard with his push mower. How many minutes will it take them working together?

6. __________ What is the units digit of the product $(3^{75})(2^{113})$?

7. __________ What is the value of the expression $\dfrac{\frac{1}{3}+\frac{1}{4}+\frac{1}{5}}{\frac{1}{2}+\frac{1}{5}+\frac{1}{6}}$? Express your answer as a common fraction.

8. __________ The product of three different positive integers is 2010. What is the maximum possible sum of the three integers?

9. __________ boxes You have a supply of boxes of volumes 1, 3, 9, 27 and 81 cubic meters. Given that the boxes must be filled completely, what is the least number of boxes that will hold exactly 300 cubic meters of sand?



10. __________ If $F(n) = 3n - 5$, $G(n) = n^2 + 3n - 2$ and $H(n) = 15 - 0.3n$, what is the value of $\dfrac{F(4) \times H(-10)}{G(-3)}$?

Workout 4

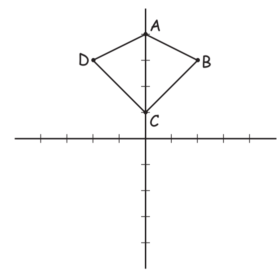
1. ________ cm³ When water freezes, its volume is increased by one-eleventh. In other words, the volume of ice equals the sum of the volume of the original amount of water and the product of one-eleventh and the volume of the water. If 979 cubic centimeters of water is to be frozen, what will be the volume of the ice that will be formed?

2. ________ meters A right triangle has an area of 84 m^2 and has integral side lengths. What is the perimeter of this triangle?

3. ________ What is the least positive integer that has a remainder of 0 when divided by 3, a remainder of 1 when divided by 4, and a remainder of 3 when divided by 7?

4. ________ If $u + w = x$ and $w + x = y$ and $x + y = z$, what is the integer value of the sum $u + w + x + y + z$, given that $u = 8$ and $z = 10$?

5. ________ What is the smallest counting number n such that the product $245n$ is a perfect square?



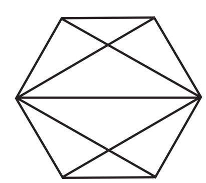
6. (,) A kite is graphed in the coordinate plane as shown. If this kite is rotated 180° clockwise about point C(0, 1), then translated down 4 units and to the left 2 units, what are the coordinates of the final image of point B(2, 3)?

7. ________ %

In 1973, a gallon of gasoline sold for 40 cents. In 2008, a gallon of gasoline sold for \$3.75. By what percent did the cost per gallon of gasoline increase? Express your answer to the nearest tenth.

8. ________ Jason was describing a spinner to his friend. He said the spinner was divided into six equal sections, each containing an integer that was not necessarily distinct from the other integers on the spinner. He told his friend that on the spinner the probability of landing on 2 is $\frac{1}{6}$, the probability of landing on 6 is $\frac{1}{6}$, the probability of landing on a factor of 27 is $\frac{2}{3}$, and the sum of the odd numbers on the spinner is 16. What is the product of the six numbers on the spinner?

9. ________ triangles Using only lines that are already drawn, how many triangles are in the regular hexagon shown?



10. ________ inches What is the length, in inches, of a rectangle that has an area of 18.75 sq in and a length that happens to be $\frac{3}{8}$ of the perimeter? Express your answer as a decimal to the nearest tenth.

Warm-Up 9

1. ________ cm^3 A certain object has a volume of 216 cubic centimeters and a surface area of 240 square centimeters. Another object is a smaller-scale replica of the original object. If the surface area of the smaller object is 60 square centimeters, what is its volume?

2. __________ Five thousand runners will participate in the Mountain Marathon. Each runner will randomly be given a different number from 1 to 5000 to wear during the race. What is the probability that the number of the second-place finisher will be both greater than the number of the third-place finisher and less than the number of the first-place finisher? (Assume no ties.) Express your answer as a common fraction.

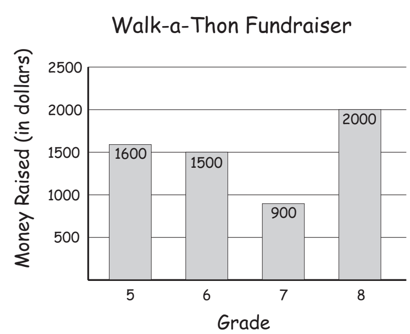
3. ______ marbles Tricia has a bag of marbles. She gave one-third of them to Barbara and one-fourth of the remaining marbles to Sam. If there are now 24 marbles in the bag, how many marbles did Tricia give to Barbara?

4. __________ A cube has a side length of 50 inches. What is the positive difference between the numerical values of the surface area of the cube in square inches and the volume of the cube in cubic inches?

5. ________ units The point P(1, 4) is reflected over the line $y = x$ to point P′. What is the distance between P and P′? Express your answer in simplest radical form.

6. __________ Think of the set of positive two-digit integers having two different digits that are selected from the digits 2, 3, 4 and 7. In this set, what is the ratio of prime numbers to composite numbers? Express your answer as a common fraction.

7. ______ degrees The students at Valley Middle School held a Walk-a-Thon to raise money for a local charity. The bar graph shows how much money was collected by each grade. If the data had been graphed in a pie chart instead, how many degrees would be in the central angle of the sector representing grade 7?



8. __________ If the seven letters of ALABAMA are to be arranged at random, what is the probability that all four As will be together? Express your answer as a common fraction.

9. ________ units^2 A trapezoid with vertices at (–1, 2), (1, 2), (3, –2) and (–3, –2) is graphed in the coordinate plane. What is the area of the region of the trapezoid where $x \geq 0$?

10. _________ What is the product of the roots of $64x + 16x^2 - 4x^3 - x^4 = 0$?

Warm-Up 10

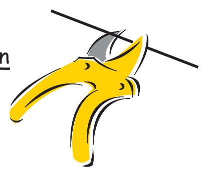
1. ______minutes Kevin can mow a lawn in 40 minutes. When Josh helps him, they can finish the lawn in 15 minutes. How long would it take Josh to mow the lawn by himself?

2. __________ In the addition problem shown, A, B, C and D represent distinct digits.

$$\begin{array}{r} ABC \\ +DBBB \\ \hline 2011 \end{array}$$

What is the value of A + B + C + D?

3. __________ What is the probability that a positive three-digit integer that has a 3 in the units place is divisible by 3? Express your answer as a common fraction.

4. ________cu in  A 96-inch piece of wire is cut into pieces, and then the pieces are to be glued together to form the edges of a box. What is the largest possible volume of a box that could have those pieces for edges?

5. ______quarts A juice drink contains only 10% juice. How many quarts of pure juice must be added to 1 gallon of the juice drink to create a drink that is 60% juice?

6. ______factors How many factors does 1800 have that are greater than 15?

7. __________ Two cubes, each with faces with 0 through 5, are to be rolled. What is the probability that the sum will be less than 6? Express your answer as a common fraction.

8. __________ The mean, median and range are all the same for a particular set of four numbers. What is the largest number in the set if the smallest number is 17?

9. _______units^2 The diagonals of a square intersect at the point (3, 2). The coordinates of one of the vertices of the square are (5, 4). What is the area of this square?

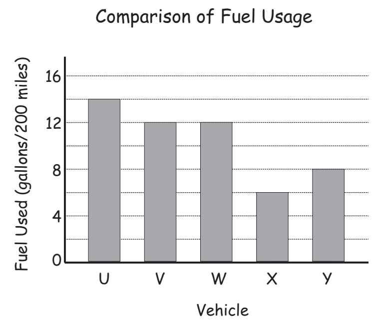
10. _________ In the multiplicative magic square shown, the product of the three numbers in each row, in each column and along each diagonal is 1. What is the value of r + s? Express your answer as a common fraction.

p	q	r
s	1	t
u	4	$\frac{1}{8}$

Workout 5

1. ________ days A painter takes two days to paint a room (all four walls and the ceiling). If he works at the same pace, how many days will it take him to paint a room that is twice as wide, twice as long, and twice as high?

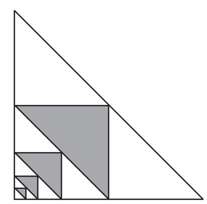
2. ________ mpg The graph shows the amount of fuel used by five different vehicles that each drove 200 miles during a road test. What is the average miles per gallon for the test? Express your answer as a decimal to the nearest tenth.



3. ________ units What is the length of the side of the only equilateral triangle whose perimeter has the same numerical value in units as its area in square units? Express your answer as a decimal to the nearest tenth.

4. $ ________ Three books and eight CDs cost twice as much as one book and five CDs. If one book and one CD cost $45, what is the cost of six books?

5. ________ units² If the largest triangle in this figure is a right isosceles triangle with an area of 1 square unit, and if all of the triangles are similar, what is the combined area of the shaded regions? Express your answer as a common fraction.



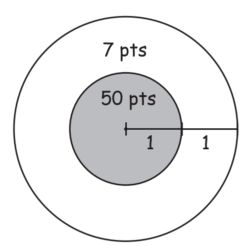
6. ________ What is the sum of the integer values of x, where $x < 8$, that produce an integer value for the expression $\sqrt{2+4+6+x}$?

7. ________ %

Assume a diamond's value is directly proportional to the square of its weight in carats. A huge diamond weighing 10 carats is broken into a 4-carat piece and a 6-carat piece. The sum of the values of the two pieces is what percent of the original value?

8. ________ A two-year study was just completed. The project began with 40 participants and ended with 90 participants. After the first year, P participants were in the study. If the percent increase of the number of participants was the same for both years, what is the value of P?

9. ________ When a dart hits within the center circular region of this circular target, the score is 50, and when a dart hits within the outer ring, the score is 7. If three darts randomly hit the target, what is the probability that the total score is 64? Express your answer as a common fraction.



10. ________ sq units A square with a diagonal of length 7 units is inscribed in a circle. What is the positive difference between the area of the square and the area of the circle? Express your answer as a decimal to the nearest hundredth.

Warm-Up 11

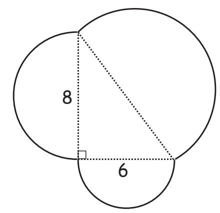
1. ________ free throws In basketball, a player can score via 3-point shots, 2-point shots and 1-point free throws. If Shakeel made eight 2-point shots and scored 30 points in all, what is the minimum number of free throws he could have made?

2. ________ m^2 The perimeter of a rectangle is 62 meters. The length is 1 meter more than four times the width. What is the area of the rectangle, in square meters?

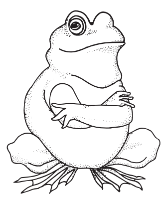
3. ________ A line segment AB has endpoints at A(5, 9) and B(-7, -15). What is the sum of the coordinates of its midpoint?

4. ________ If P is divided by R, the result is $\frac{2}{3}$. If R is divided by S, the result is $\frac{3}{4}$. What is the result if P is divided by S? Express your answer as a common fraction.

5. ________ $units^2$ The region shown is composed of a right triangle and three semicircles. What is the area of the entire enclosed region if the triangle has legs of 6 and 8 units? Express your answer in terms of π.



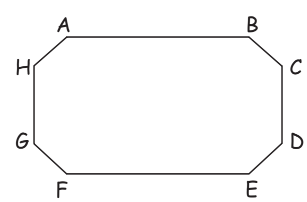
6. ________



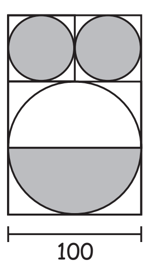
Julie collects toy frogs. She displays them on a shelf and rearranges them often. If there are 3 green frogs and 2 yellow frogs lined up on her shelf, what is the probability that the frogs on both ends will be green? Express your answer as a common fraction.

7. ________ What is the product of all integer values of x for which $|x^2 - 9|$ is a prime number?

8. ________ Convex octagon ABCDEFGH has congruent angles A, B, E and F and congruent angles C, D, G and H. If $m\angle A = (2x)°$, $m\angle B = y°$ and $m\angle C = (x + 50)°$, what is the value $x + y$?



9. ________ $units^2$



A square of width 100 units contains a circle tangent to all four sides of the square. Two smaller squares, congruent to each other and each containing a circle tangent to all four sides of the square, rest atop the large square, with the three squares together forming a rectangle. What is the difference between the area of the shaded semicircle and the combined area of the two smaller shaded circles?

10. ________ ways Two subsets of the set S = {a, b, c, d, e} are to be chosen so that their union is S and their intersection contains exactly three elements. Each of the elements of S is distinct. In how many ways can this be done, assuming that the order in which the subsets are chosen is irrelevant?

Warm-Up 12

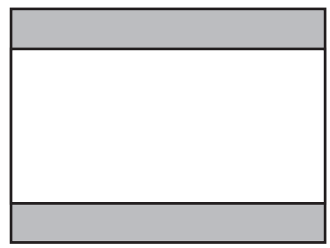
1. __________ The numerator of a fraction is 8 less than the denominator. If the numerator is doubled and the denominator is decreased by 1, the value of the resulting fraction is 1. What is the original fraction? Express your answer as a common fraction.

2. (,) Point A has coordinates (-2, 5) and point B has coordinates (10, 8). Point P is on segment AB, with AP = 2PB. What are the coordinates of point P? Express your answer as an ordered pair.

3. __________ What is the value of x if $5^2(5) + 5^2(11) + 5^2(17) + 5^2(78) + 5^2(14) = 5^x$?

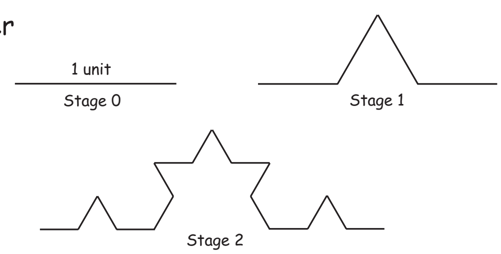
4. __________ The product of two consecutive, positive, odd integers is 783. What is the sum of the two integers?

5. __________ % The aspect ratio of a rectangle is its length divided by its width. The figure shows an image (the white rectangle) with aspect ratio 16:9 displayed on a screen (the largest rectangle) with aspect ratio 4:3. What percent of the area of the screen is occupied by the image?

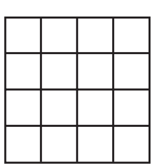


6. __________ The mean of the squares of three non-zero numbers a, b and c is three times the square of their mean. If $b = 2a$, what is the value of $\frac{c}{a}$? Express your answer as a common fraction.

7. __________ units longer When creating a particular pattern, the designer begins with a line segment 1 unit long. Each successive stage is created by using congruent segments that are half as long as those in the previous stage. If you trace the design at Stage 4, how many units longer is it than the length of the design at Stage 1?



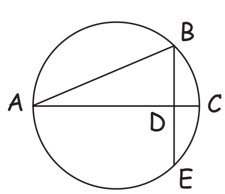
8. __________



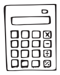
Two different unit squares are randomly selected from the 16 unit squares in the 4 × 4 grid shown. What is the probability that they do not have a vertex in common? Express your answer as a common fraction.

9. __________ The graphs of $y = x^2 - 8x - 35$ and $y = -2x^2 + 16x + 3$ intersect in two points. What is the sum of the x-coordinates of the two points of intersection?

10. __________ units



In the figure shown, $\overline{AC}$ is the diameter of the circle. If the measure of arc BC is 60°, $\overline{AC} \perp \overline{BE}$, and BE = 12 units, what is the length of $\overline{AB}$?



Workout 6

1. ________ cents Two apples, five bananas and one carrot cost a total of \$2.05. Three apples, one banana and four carrots cost a total of \$1.89. Three apples, two bananas and three carrots cost a total of \$1.98. What is the total cost, in cents, of one apple, one banana and one carrot?

2. ________ points In math class, the mean test score for the girls was 92 points and the mean test score for the boys was 87 points. If 12 girls and 10 boys took the test, what was the mean test score for the whole class? Express your answer as a decimal to the nearest hundredth.

3. ________ % Tria paid \$11,500 for a new car after the original price was reduced by a rebate of \$4500. What percentage of the original price did Tria pay for the car? Express your answer to the nearest tenth.

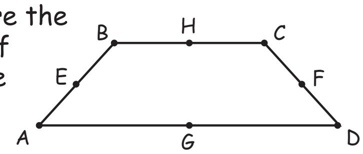
4. ________ miles

How many miles can Scotty ride on a bike, going at the rate of 8 miles per hour, if he must walk back to the starting point at a rate of 3 miles per hour (following the same route he traveled on his bike) and he is to be gone a total of 11 hours?

5. ________ inches^3 In order to send his DVD player back to the manufacturer for repair, Jeremy must pack it securely. He has a packing box with internal dimensions 22″ × 18″ × 6″. The DVD player is a rectangular prism with dimensions 15″ × 12″ × 2″. He has a box with external dimensions 3″ × 3″ × 8″ for the power cord, which also will go inside the packing box. How many cubic inches of packing material must he put in the packing box to fill it completely?

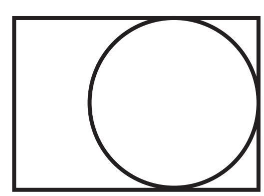
6. \$________ Stella sells 30 gismos a month at a price of \$20 per gismo. She notices that each time she raises the price of a gismo by \$4, she sells 2 fewer gismos per month. What is the maximum amount of revenue, in dollars, that Stella can make in a month of selling gismos?

7. ________ units^2 Isosceles trapezoid ABCD has bases with lengths 12 and 24. Points E and F are the midpoints of legs AB and CD, respectively. Points G and H are the midpoints of bases AD and BC, respectively. If the height of the trapezoid is 6 units, what is the difference between the areas of trapezoid ABCD and parallelogram HFGE?



8. ________ feet Mr. Kelly is building a 60-foot-long ramp that will be 24 feet high at the highest point. He wants to place a vertical support beam under the ramp with its top 15 feet down the length of the ramp. How long does he need to make the support beam?

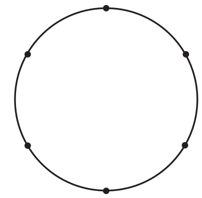
9. ________ % A circular garden is placed inside a rectangular patio that measures 20 feet by 14 feet. The circular garden is tangent to three sides of the rectangular patio, as shown. What percent of the patio is outside of the garden? Express your answer to the nearest whole number. Use $\frac{22}{7}$ as an approximation for π.



10. ________ A bag contains five red marbles, three blue marbles and two green marbles. If six marbles are drawn without replacement from the bag, what is the probability that two marbles of each color are drawn? Express your answer as a common fraction.

Warm-Up 13

1. __________ Six points are equally spaced around a circle. If three of the points are selected at random, what is the probability that they form a scalene triangle? Express your answer as a common fraction.



2. __________ Point P is on the y-axis and is equidistant from the points (1, 3) and (7, 5). What is the y-coordinate of point P?

3. ________ am

A manufacturing plant periodically opens a large valve at the top of a tank to pump water into the tank for use in a manufacturing process. When the valve is opened, water is pumped into the tank at a rate of 75 gallons per minute. Another valve, at the bottom of the tank, sends water out of the tank to an assembly line at a rate of 24 gallons per minute. Both valves are opened simultaneously at 7:30 am and begin filling the empty tank. At what time, to the nearest minute, will the tank be filled to its 600-gallon capacity?

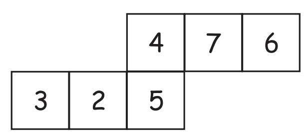
4. __________ A set of 5 positive integers has a mean of 11, a median of 10 and a unique mode of 7. What is the greatest possible range of the set of integers?

5. __________ A jar contains $5.00 in quarters, $5.00 in dimes, $5.00 in nickels and $5.00 in pennies. If Connie randomly chooses one coin from the jar, what is the probability that the coin chosen is a dime? Express your answer as a common fraction.

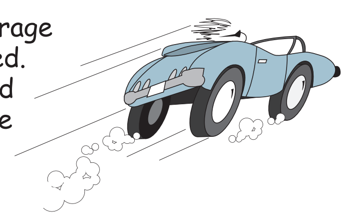
6. __________ Marketing analysts are investigating cylindrical packaging for a new beverage called Star Juice. One member of the design team proposed a can that has a diameter of 6 cm and a height of 20 cm. Another suggested a can that is 17 cm tall and has a diameter of 8 cm. What is the ratio of the smaller volume to the larger volume? Express your answer as a common fraction.

7. ________ values For how many integer values of x is the following inequality true? $-5 \le 2x - 7 \le 21$

8. __________ The figure shown is a net of a cube. When folded to form a cube, what is the largest sum of two numbers on opposite faces?



9. ________ miles Yesterday Alice drove one hour longer than Bob at an average speed that was five miles per hour faster than Bob's speed. Clark drove two hours longer than Bob at an average speed that was ten miles per hour faster than Bob's speed. Alice drove 50 miles more than Bob. How many more miles than Bob did Clark drive?



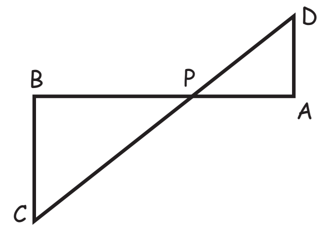
10. ________ units In right triangle ABC, the median to the hypotenuse has length 15 units and the altitude to the hypotenuse has length 12 units. What is the length of the shorter leg of triangle ABC? Express your answer in simplest radical form.

Warm-Up 14

1. ________ In base b, $321 - 123 = 154$. What is the value of b?

2. ________ What is the value of n if $4(3^{n+2}) + 45(3^n) = 3$?

3. ________ cm In the figure shown, AD = 4 cm, BC = 8 cm and CD = 20 cm. Also, $\overline{AD} \perp \overline{AB}$ and $\overline{BC} \perp \overline{AB}$. What is the length of $\overline{AP}$? Express your answer as a common fraction.



4. ________ A rectangle with sides of length 5 units and 6 units that are parallel to the x-axis and y-axis has one vertex at (1, 3). What is the largest possible sum of the coordinates of any of the other vertices of this rectangle?

5. ________ What is the units digit of N in the following equation? $\dfrac{3 \times 3 \times 3 \times 3 \times 3}{N} = \dfrac{1}{2} \times \dfrac{1}{3} \times \dfrac{1}{3}$

6. ________ tricycles The cycle shop sells only bicycles and tricycles. All the bicycles have two wheels and all the tricycles have three wheels. If you add up the total wheels on the 67 cycles in the shop, you get 157 wheels. How many tricycles are in the shop?

7. ________ cm^2 If ABCDEFGH is a regular octagon with sides of length 2 cm, what is the area, in square centimeters, of triangle ABC? Express your answer in simplest radical form.

8. ________ minutes

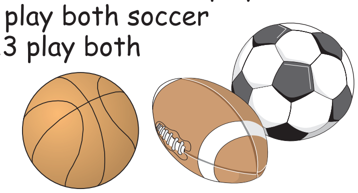
Running at 12 miles per hour burns about 1200 calories per hour. Running at 5 miles per hour burns about 450 calories in an hour. How many minutes longer does one have to run to burn 300 calories at 5 miles per hour than at 12 miles per hour?

9. ________ students Of the 2000 students in Houston High School, 70% are male. How many male students must be moved to another school so that 60% of the Houston High student body will be males?

10. ________ Suppose that six brothers (George, John, Thomas, Fred, Andrew, Martin) each roll a fair 6-sided die. At least three of them have rolled an even number. What is the probability that John, Fred and Martin have each rolled an even number? Express your answer as a common fraction.

Workout 7

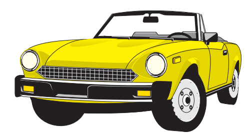
1. ________ students At Marsh Mel Low Middle School 34 students play soccer, 36 students play football and 29 students play basketball. Of these students 15 play both soccer and football, 18 play both basketball and football and 13 play both soccer and basketball. What is the smallest possible number of students that play all three sports?



2. ________ If x and y are two triangular numbers less than 100 that when added, produce a sum that is a square number, what is the largest possible value of x or y?

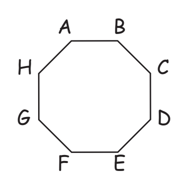
3. ________ gallons Tria's new car can get 50% more miles per gallon of gasoline than her old car could. If her new car gets 24 miles per gallon, how many gallons of gasoline will she save using her new car to travel 480 miles to visit relatives in comparison with driving the old car the same distance? Express your answer to the nearest whole number.

4. ________ hrs Mr. Davidson lives 30 miles from his office. Driving to work on Monday, he averaged 50 mph. However, during his return trip from work, there was construction that slowed traffic. Mr. Davidson averaged only 20 mph going home. How many hours did Mr. Davidson spend traveling to and from work on Monday? Express your answer as a decimal to the nearest tenth.



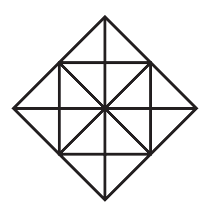
5. ________ Exactly one of the following statements is true. Which one?
A. Exactly one of these statements is false.
B. Exactly two of these statements are false.
C. Exactly three of these statements are false.
D. Exactly four of these statements are false.
E. Exactly five of these statements are false.

6. ________ In regular octagon ABCDEFGH, what is the ratio of the area of triangle ADF to the area of triangle AHF? Express your answer as a decimal to the nearest hundredth.



7. $ ________ If $1000 is invested at 6% annual interest compounded semiannually, how much interest has been earned at the end of one year?

8. ________ rect How many rectangles are there in the following figure?



9. ________ ways Four children (Alice, Brad, Cathy and Dan) are arranged in a line. If Brad and Cathy cannot be next to each other, in how many ways can the four children be arranged?

10. ________ units² In the coordinate plane, what is the area of the region that is above the x-axis, between $x = -3$ and $x = 3$, and below $y = |||x|-1|-2|$?

Warm-Up 15

1. ________ packages

Fantasy gum is sold in 3-ounce and 10-ounce packages. One pound is equal to 16 ounces. If 76 packages were sold at B-Rite Mart last month and the total weight sold was less than 16 pounds, what is the greatest number of 10-ounce packages B-Rite could have sold?

2. __________ Three standard, six-sided dice are rolled. The sum of the three numbers on top is less than 6. What is the probability that all three dice show a 1? Express your answer as a common fraction.

3. __________ A line through point P(4, 3) is parallel to the line $x - 2y = 8$. What is the equation of the line through point P? Express your answer in slope-intercept form where the slope is a common fraction.

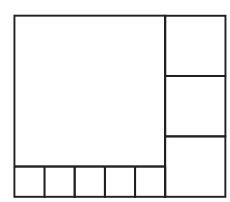
4. ________ ways How many different ways can the letters in the name HANK be scrambled so that the newly created "words" start and end with a consonant and are not HANK?

5. ________ sub-committees Eight sophomores are on a student council committee. A party planning subcommittee must be selected from these eight student council committee members. How many different party planning subcommittees are possible if it must have more than 1 but less than 5 members?

6. ________ pairs A regular hexagon and a regular octagon have the same perimeter. If the side lengths of these polygons are whole numbers and the sum of the perimeters is less than 200, how many different pairs of hexagons and octagons can be made?

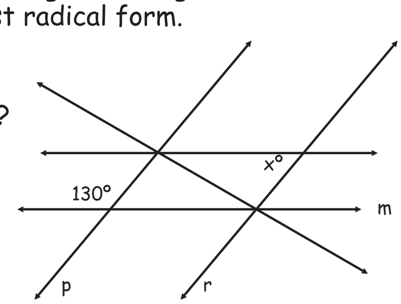
7. __________ What is the integer closest to $\sqrt{150} + \sqrt{75}$?

8. ________ units² A rectangle has been partitioned into nine squares, as shown. If the area of each of the three medium squares is 100 square units, what is the area of the largest square?



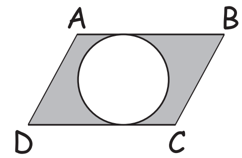
9. ________ cm² A circle with a radius of $4\sqrt{3}$ cm is inscribed inside a regular hexagon. What is the area of the hexagon? Express your answer in simplest radical form.

10. __________ In the figure, $l \parallel m$ and $p \parallel r$. What is the value of x?



Warm-Up 16

1. __________ units^2 In this figure, side AB of parallelogram ABCD is 10 units long. A circle of radius 3 units is drawn in the interior such that sides AB and CD of the parallelogram are tangent to it. What is the area of the shaded region? Express your answer in terms of π.



2. __________ If $f(x) = 5x$, and $g(x) = f(x) - 3x - 7$, what is the value of $g(8)$?

3. __________ The sum of the digits of a two-digit positive integer is 8. When the digits are reversed, the new integer is 3 more than 4 times the original integer. What is the new integer?

4. __________ degrees The largest angle of a quadrilateral is four times its smallest angle. Another angle is 10 degrees more than twice the smallest. The fourth angle is 60 degrees less than 3 times the smallest. What is the measure of the largest angle?

5. __________ students At Melville Middle School 73 students are in the band, 65 students are in the chorus and 114 students play sports. Thirty-two are both in the band and play sports, 12 play sports and sing in the chorus, but no one is in both the band and the chorus. Five hundred fifty-eight students do not participate in any of these activities. How many students are there at Melville Middle School?

6. __________ sides How many sides does a convex polygon with 20 diagonals have?

7. __________ % On the number line, what percent of the interval $-10 \le x \le 10$ satisfies the inequality $x + 2 < \dfrac{5}{x-2}$?

8. __________ If $a \,\Delta\, b = \dfrac{a-b}{2b+6}$, what is the value of $3 \,\Delta\, (5 \,\Delta\, 3)$? Express your answer as a common fraction.

9. __________ integers How many integers n satisfying $3000 \le n \le 4000$ have the product of their digits equal to zero?

10. __________ Let PQRS be an isosceles trapezoid with bases PQ = 100 units and RS = 26 units. Suppose PS = QR = x units and a circle with center on base PQ is tangent to both segments PS and QR. If x is the smallest possible value, then what is the value of x^2?

Workout 8

1. ______ inches Each of Jonathan's bicycle tires revolved 2327 times as he rode three miles. What is the diameter of one of Jonathan's bicycle tires, in inches? Express your answer to the nearest whole number.

2. ______ times Six friends A, B, C, D, E and F eat dinner at a local restaurant. A eats there every day. B eats there every other day. C eats there every third day. D eats there every fourth day. E eats there every fifth day. F eats there every sixth day. If all six of them eat together at the restaurant on January 15, how many more times will they eat together in that calendar year?

3. ______ liters The kettle in Kerry's kitchen is 80% full of water. After 25% of the water in it has been poured out, there are 1200 ml of water left. How many liters of water does Kerry's kettle hold when it is full?

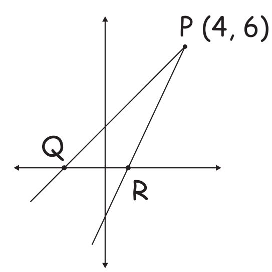
4. ______ inches^2 A square is inscribed in a circle with radius 6 inches. What is the area inside the circle but outside the square? Express your answer to the nearest whole number.

5. ______ % Next week, to celebrate 50 years in business, the grocery store is giving away a bag of free groceries to every 150th customer, starting with the 150th customer. If 2000 customers shop at the store next week, what is the percentage of customers who will receive a free bag of groceries? Express your answer to the nearest hundredth.

6. $ ______ Tom buys two shirts and one pair of socks for $15. Sam buys one shirt and two pairs of socks for $12. How much will Jim pay for 15 shirts and 10 pairs of socks?

7. ______ feet In isosceles triangle ABC, AB = AC = 13 ft, and BC = 10 ft. Square RSTU is inscribed in the triangle, with R and S on $\overline{BC}$, with T on $\overline{AC}$ and with U on $\overline{AB}$. What is the length of a side of the square? Express your answer as a common fraction.

8. ______ units^2 A line with slope equal to 1 and a line with slope equal to 2 intersect at the point P(4, 6), as shown. Points Q and R are both on the *x*-axis. What is the area of triangle PQR?



9. ______ When Andy, Becky, Chad, Dave, Eleanor and Fred line up in a row, what is the probability that Andy and Becky are not standing next to each other? Express your answer as a common fraction.

10. ______ units^2 In convex pentagon JKLMN, KL = LM = MN = 4 units and JK = JN = 8 units. If $m\angle J = 60^\circ$ and $m\angle L = m\angle M$, what is the area of the pentagon, expressed in simplest radical form?

Warm-Up 17

1. __________ days For every day his homework is late, Sam's grade is reduced by 10%. For example, if Sam's homework is 2 days late, his score will be multiplied by 0.8. Sam will get 4 more problems correct for each extra day he works on his homework past the due date. If Sam has 14 out of 30 problems correct the day his assignment is due, how many days late should he turn it in to get the maximum grade?

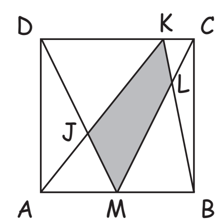
2. __________ ways Eight identical marbles are to be placed in five boxes numbered 1 through 5 so that each box contains at least one marble. In how many ways can this be done?

3. __________ Two natural numbers have a greatest common factor of 3 and a least common multiple of 216. If the difference between the two numbers is 3, what is the sum of the two numbers?

4. __________ The table lists points (x, y) that satisfy the linear relation $y = mx + b$. What is the product of m and b? Express your answer as a common fraction.

x	1	5	9	13
y	2	5	8	11

5. __________ units² Square ABCD has sides of length 4 units. M is the midpoint of $\overline{AB}$, and K is on $\overline{CD}$ so that DK:KC = 3:1. $\overline{DM}$ and $\overline{AK}$ intersect at J. $\overline{MC}$ and $\overline{BK}$ intersect at L. What is the area of quadrilateral JKLM? Express your answer as a common fraction.



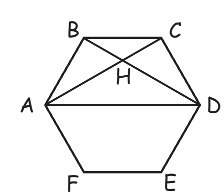
6. __________ Given that $n > 1$, what is the smallest positive integer n whose divisors have a product of n^8?

7. __________ units² In triangle RST, J is on line segment RT, with RJ:JT = 2:1. Also, K is on line segment ST, with TK:KS = 2:1. Line segments SJ and RK intersect at point P. If the area of triangle SPK is 7 units², what is the area of triangle RPS?

8. __________ minutes At 3:00, the hour hand and minute hand of a 12-hour clock are perpendicular. What is the least number of minutes that must elapse for this to be true again? Express your answer as a mixed number.

9. __________ balls One hundred red balls are lined up in a row. Starting from the left end, every fourth ball is replaced with a green ball. Then, starting from the right end, every fifth ball is replaced with a white ball. Finally, starting from the left end, every sixth ball is replaced with a yellow ball. How many red balls remain in the row?

10. __________ If ABCDEF is a regular hexagon, what is the ratio of the area of triangle AHD to the area of hexagon ABCDEF? Express your answer as a common fraction.

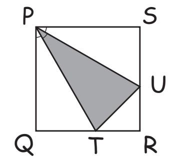


Warm-Up 18

1. __________ What is the product of x and y if $x^2 + y^2 = 36 - 2xy$ and $x^2 - y^2 = 12$?

2. __________ For each positive two-digit integer, John adds the two digits. For example, 34 gives $3 + 4 = 7$. What is the sum of all of his results?

3. ________ units^2 The figure shows ΔPTU inscribed in square PQRS. If PQ = 12 units and segments PT and PU trisect $\angle QPS$, what is the area of ΔPTU?



4. __________ The mean, median and range are all 6 for a collection of five positive integers. What is the sum of all possible distinct values that could be the greatest number in all the possible collections?

5. __________ What is the sum of the reciprocals of the roots of $2x^2 + 3x - 8 = 0$? Express your answer as a common fraction.

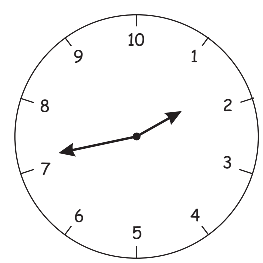
6. __________ This Sudoku-like figure is a 4×4 grid to be filled so that each of the digits 1, 2, 3 and 4 appears in each row and in each column. The 4×4 grid is divided into four 2×2 squares. Each of these 2×2 squares also is to contain each of the digits 1, 2, 3 and 4. What digit replaces P?

1		3	
	2		
	P		
			4

7. __________ A small square is inscribed in a circle that has another, larger square circumscribed about it. What is the ratio of the area of the small square to the area of the large square? Express your answer as a common fraction.

8. ________ units^2 Three rectangles have the same area. Their dimensions are m by n, $(m + 2)$ by $(n - 2)$ and $(m - 2)$ by $(n + 10)$. What is the area of each rectangle?

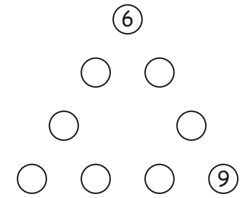
9. ________ minutes Ben came up with a new idea for a clock and telling time. He wants to measure a day as two 10-"hour" cycles rather than two 12-hour cycles. However, Ben does not want to change the duration of a minute, so a day would still have the same number of minutes. How many minutes are there in a "Ben-clock hour"?



10. ________ cubes Two red, two yellow and two green faces, all unit squares, are available for building a cube. How many distinct cubes can be built?

Workout 9

1. __________ Each of the integers 1 through 9 is to be placed in one of the nine given circles so that the sum, S, of the four numbers along each side of the triangle is the same. Two of the numbers have been entered. What is the value of S?



2. ________ units In trapezoid ABCD, bases AB and CD have lengths 10 and 6, respectively. Line segment PQ is drawn parallel to the bases, with P on $\overline{AD}$ and Q on $\overline{BC}$ so that the area of ABQP is three-fourths that of ABCD. What is the length of PQ? Express your answer in simplest radical form.

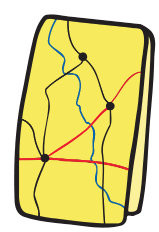
3. ________ ways The number 20 can be expressed as a sum of three natural numbers in many ways. Three distinct examples are 1 + 12 + 7, 3 + 14 + 3 and 14 + 3 + 3. Including the examples shown, in how many distinct ways can 20 be expressed as a sum of three natural numbers?

4. ________ mm^3 Base BCD of right tetrahedron ABCD is an equilateral triangle with sides of length 6 mm. Each of the lateral sides of the tetrahedron is an isosceles right triangle. What is the volume of the tetrahedron? Express your answer in simplest radical form.

5. ________ numbers Of the first 2011 natural numbers, how many have exactly three digits of 1 when written in base 2 form?

6. ________ cm^2 Each side of equilateral triangle ABC has length 60 cm. Point U is on $\overline{AB}$, with AU:UB = 1:2; point V is on $\overline{BC}$ with BV:VC = 1:3; and point W is on $\overline{AC}$, with CW:WA = 1:4. What is the area of triangle UVW? Express your answer in simplest radical form.

7. ________ miles On Sunday, John drove from his house to his uncle's house for a visit. If his average speed had been 10 miles per hour slower, the trip would have taken 2 hours longer. If his average speed had been 20 miles per hour faster, the trip would have taken 2 hours less. How many miles is it from John's house to his uncle's house?

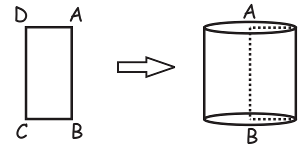


8. __________ If a positive two-digit integer is c times the sum of its digits, the number formed by interchanging the digits is the sum of the digits multiplied by what expression that involves c?

9. ________ ordered pairs How many ordered pairs of integers (x, y) satisfy $x^2 + y^2 = 65^2$?

10. __________ There are several natural numbers, n, such that $B = n^2 + 2008n$ and B has a units digit of 4. What is the largest possible digit that can be in the tens place of B?

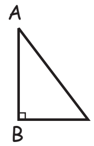
Solids Stretch



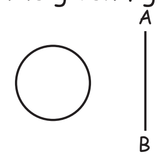
If you revolve rectangle ABCD about side AB, you will generate a cylinder.
Line AB is the axis of revolution for this solid of revolution.

For questions 1 and 2, describe the solid of revolution generated by revolving the given figure 360° about segment AB.

1. ____________

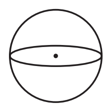


2. ____________

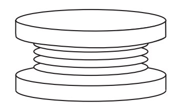


For questions 3 and 4, what plane figure will generate the given solid when revolved 360° about an axis of revolution? Draw or describe the figure and indicate the axis of revolution.

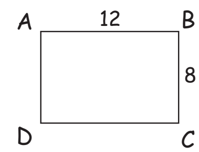
3. ____________



4. ____________



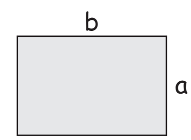
5. If you revolve this rectangle about side AB or side BC, you form a cylinder.



a. Revolving around which of these two segments produces a cylinder with a larger volume? _________

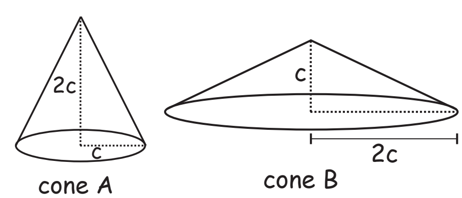
b. Revolving around which of these two segments produces a cylinder with a larger total surface area? ________

6. _________ Consider this shaded rectangle with b > a. Let V_a denote the volume of the cylinder that is created when the shaded rectangle is revolved around side a, and let V_b denote the volume of the cylinder that is created when the shaded rectangle is revolved around side b. What is the value of the ratio $\frac{V_b}{V_a}$? Express your answer as a common fraction in terms of a and b.

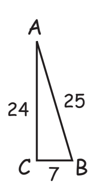


7. _________ Let S_a denote the total surface area of the cylinder that is created when the shaded rectangle used in question 6 is revolved around side a, and let S_b denote the total surface area of the cylinder that is created when the shaded rectangle is revolved around side b. What is the value of the ratio $\frac{S_b}{S_a}$? Express your answer as a common fraction in terms of a and b.

8. _________ These cones can be formed by revolving a right triangle about its two legs. What is the ratio of the volume of cone A to the volume of cone B? Express your answer as a common fraction.



9.



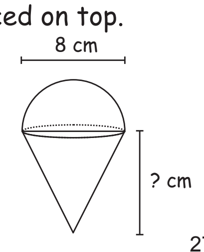
The side lengths in the figure are given. What is the volume of the solid that results from revolving the triangle 360° around . . .

a) side AC? Express your answer in terms of π. __________ $units^3$

b) side BC? Express your answer in terms of π. __________ $units^3$

c) side AB? Express your answer as a common fraction in terms of π. __________ $units^3$

10. An ice cream cone is packed full of ice cream, and a hemisphere of ice cream is placed on top.



a. What plane figure will generate this solid when revolved 360° about an axis of revolution? Draw or describe the figure and indicate the axis of revolution. ______

b. If the volume of ice cream inside the cone is the same as the volume of ice cream outside the cone, how many centimeters is the height of the cone? __________ cm

Sum and Product SUPER Stretch

1. __________ What is the sum of the solutions of $6x^2 + 5x - 4 = 0$? Express your answer as a common fraction.

2. __________ A quadratic equation of the form $x^2 + kx + m = 0$ has solutions $x = 3 + 2\sqrt{2}$ and $x = 3 - 2\sqrt{2}$. What is the value of $k + m$?

3. __________ What is the sum of the reciprocals of the solutions of $4x^2 - 13x + 3 = 0$? Express your answer as a common fraction.

4. __________ If r and s are the solutions of $2x^2 + 9x + 3 = 0$, what is the value of $r^2 + s^2$? Express your answer as a common fraction.

5. __________ If r and s are the solutions of $x^2 + 6x - 2 = 0$, what is the value of $r^3 + s^3$?

6. (___ , ___) The solutions of $x^2 + bx + c = 0$ are each 5 more than the solutions of $x^2 + 7x + 3 = 0$. What are the values of b and c? Express your answer as an ordered pair (b, c).

7. (___ , ___ , ___) A cubic equation of the form $x^3 + bx^2 + cx + d = 0$ has solutions $x = 3$, $x = 4$ and $x = 5$. What are the values of b, c and d? Express your answer as an ordered triple (b, c, d).

8. __________ What is the sum of the reciprocals of the solutions of $x^3 - 3x^2 - 13x + 15 = 0$? Express your answer as a common fraction.

9. __________ What is the sum of the squares of the solutions of $x^3 - 15x^2 + 66x - 80 = 0$?

10. __________ The solutions of $x^3 - 63x^2 + cx - 1728 = 0$ form a geometric sequence. What is the value of c?

ANSWERS TO HANDBOOK PROBLEMS

In addition to the answer, we have provided a difficulty rating for each problem. Our scale is 1-7, with 7 being the most difficult. These are only approximations, and how difficult a problem is for a particular student will vary. Below is a general guide to the ratings:

Difficulty 1, 2 and 3 - One concept; one- to two-step solution; appropriate for students just starting the middle-school curriculum.
4 and 5 - One or two concepts; multistep solution; knowledge of some middle-school topics is necessary.
6 and 7 - Multiple and/or advanced concepts; multi-tep solution; knowledge of advanced middle-school topics and/or problem-solving strategies is necessary.

Warm-Up 7

Answer	Difficulty				
1. 8700	(2)	4. $\frac{31}{32}$	(2)	7. 11111	(3)
2. -2	(3)	5. 45	(2)	8. 36	(5)
3. 20	(2)	6. -1	(3)	9. 0	(4)
				10. 56	(2)

Warm-Up 8

Answer	Difficulty				
1. 4.75	(3)	4. $\frac{5}{24}$	(3)	7. $\frac{47}{52}$	(3)
2. 20,002	(3)	5. 75	(4)	8. 1008	(3)
3. 36	(4)	6. 4	(4)	9. 6	(2)
				10. -63	(5)

Workout 4

Answer	Difficulty				
1. 1068	(3)	4. 26	(4)	7. 837.5	(3)
2. 56	(4)	5. 5	(2)	8. 972	(3)
3. 45	(3)	6. (-4, -5)	(3)	9. 16	(4)
				10. 7.5	(5)

Warm-Up 9

Answer	Difficulty				
1. 27	(5)	4. 110,000	(4)	7. 54	(3)
2. $\frac{1}{6}$	(4)	5. $3\sqrt{2}$	(4)	8. $\frac{4}{35}$	(5)
3. 16	(3)	6. $\frac{5}{7}$	(3)	9. 8	(4)
				10. 0	(6)

Warm-Up 10

Answer	Difficulty				
1. 24	(4)	4. 512	(3)	7. $\frac{7}{12}$	(3)
2. 16	(3)	5. 5	(4)	8. 51	(5)
3. $\frac{1}{3}$	(3)	6. 25	(3)	9. 16	(4)
				10. $\frac{9}{16}$	(3)

Workout 5

Answer	Difficulty				
1. 8	(4)	4. 180 or 180.00	(4)	7. 52	(5)
2. 19.2	(3)	5. $\frac{85}{256}$	(4)	8. 60	(3)
3. 6.9	(5)	6. -30	(4)	9. $\frac{27}{64}$	(5)
				10. 13.98	(4)

Warm-Up 11

Answer	Difficulty				
1. 2	(2)	4. $\frac{1}{2}$	(3)	7. 64	(5)
2. 150	(4)	5. $24 + 25\pi$ or $25\pi + 24$	(5)	8. 220	(4)
3. -4	(2)	6. $\frac{3}{10}$	(5)	9. 0	(4)
				10. 20	(5)

Warm-Up 12

Answer	Difficulty				
1. $\frac{7}{15}$	(4)	4. 56	(3)	7. 14	(5)
2. (6, 7)	(3)	5. 75	(4)	8. $\frac{13}{20}$	(6)
3. 5	(4)	6. $-\frac{2}{3}$	(5)	9. 8	(6)
				10. 12	(5)

Workout 6

Answer **Difficulty**

1.	74	(4)
2.	89.73	(3)
3.	71.9	(3)
4.	24	(4)
5.	1944	(3)
6.	800 or 800.00	(4)
7.	54	(5)
8.	18	(4)
9.	45	(5)
10.	$\frac{1}{7}$	(6)

Warm-Up 13

Answer **Difficulty**

1.	$\frac{3}{5}$	(4)
2.	16	(6)
3.	7:42	(3)
4.	13	(3)
5.	$\frac{5}{67}$	(3)
6.	$\frac{45}{68}$	(5)
7.	14	(5)
8.	10	(3)
9.	110	(7)
10.	$6\sqrt{5}$	(6)

Warm-Up 14

Answer

1.	6	(6)
2.	-3	(5)
3.	$\frac{16}{3}$	(5)
4.	15	(3)
5.	4	(4)
6.	23	(4)
7.	$\sqrt{2}$	(6)
8.	25	(4)
9.	500	(5)
10.	$\frac{4}{21}$	(7)

Workout 7

Answer

1.	2	(5)
2.	91	(5)
3.	10	(5)
4.	2.1	(4)
5.	D	(3)
6.	2.41	(6)
7.	60.90	(4)
8.	18	(4)
9.	12	(4)
10.	7	(5)

Warm-Up 15

Answer	Difficulty				
1. 3	(3)	4. 11	(4)	7. 21	(5)
2. $\frac{1}{10}$	(4)	5. 154	(5)	8. 625	(6)
3. $y = \frac{1}{2}x + 1$	(5)	6. 4	(4)	9. $96\sqrt{3}$	(5)
				10. 50	(4)

Warm-Up 16

Answer	Difficulty				
1. $60 - 9\pi$	(4)	4. 164	(4)	7. 40	(6)
2. 9	(5)	5. 766	(5)	8. $\frac{17}{38}$	(4)
3. 71	(5)	6. 8	(4)	9. 272	(4)
				10. 1850	(7)

Workout 8

Answer	Difficulty				
1. 26	(5)	4. 41	(4)	7. $\frac{60}{11}$	(5)
2. 5	(4)	5. 0.65	(4)	8. 9	(5)
3. 2	(4)	6. 120 or 120.00	(5)	9. $\frac{2}{3}$	(6)
				10. $28\sqrt{3}$	(6)

Warm-Up 17

Answer	Difficulty				
1. 3	(4)	4. $\frac{15}{16}$	(5)	7. 42	(7)
2. 35	(6)	5. $\frac{56}{15}$	(7)	8. $32\frac{8}{11}$	(6)
3. 51	(4)	6. 120	(7)	9. 54	(6)
				10. $\frac{2}{9}$	(6)

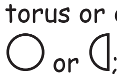
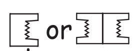
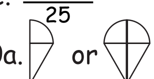
Warm-Up 18

Answer	Difficulty
1. 8	(6)
2. 855	(5)
3. 48	(6)
4. 27	(5)
5. $\frac{3}{8}$	(6)
6. 1	(4)
7. $\frac{1}{2}$	(5)
8. 15	(6)
9. 72	(4)
10. 6	(7)

Workout 9

Answer	Difficulty
1. 21	(5)
2. $2\sqrt{13}$	(6)
3. 171	(5)
4. $9\sqrt{2}$	(6)
5. 165	(6)
6. $375\sqrt{3}$	(7)
7. 240	(5)
8. $11 - c$ or $-c + 11$	(7)
9. 36	(4)
10. 8	(6)

Solids Stretch

Answer	Difficulty
1. cone	(2)
2. torus or donut	(3)
3.  ; axis contains diameter. Answers may vary.	(3)
4.  ; axis is the longest straight vertical line. Answers may vary.	(3)
5a. segment BC	(5)
b. segment BC	(5)
6. $\frac{a}{b}$	(6)
7. $\frac{a}{b}$	(6)
8. $\frac{1}{2}$	(5)
9a. 392π	(4)
b. 1344π	(4)
c. $\frac{9408\pi}{25}$	(6)
10a.  Answers may vary.	(4)
b. 8	(3)

Sum and Product Super Stretch

Answer	Difficulty
1. $-\frac{5}{6}$	(4)
2. -5	(4)
3. $\frac{13}{3}$	(4)
4. $\frac{69}{4}$	(5)
5. -252	(5)
6. (-3, -7)	(5)
7. (-12, 47, -60)	(6)
8. $\frac{13}{15}$	(6)
9. 93	(6)
10. 756	(7)

SOLUTIONS TO HANDBOOK PROBLEMS

The solutions provided here are only *possible* solutions. It is very likely that you or your students will come up with additional—and perhaps more elegant—solutions. Happy solving!

Warm-Up 7

Problem 1. Melinda should choose the least three-digit positive integer, which is 100. She will get $(3000 - 100) \times 3 = 2900 \times 3 =$ **8700**.

Problem 2. After the first term of 2, the sequence simply alternates between 1 and −2. The 2011th term is **−2**.

Problem 3. It will take Susan $480 \times 600 \div 240 = 1200$ minutes to read the book. That's $1200 \div 60 =$ **20** hours.

Problem 4. There are $2^5 = 32$ possible ways that the 5 flips can land. Only one of these is 5 heads in a row, so the probability is **31/32** that the result will not be 5 heads in a row.

Problem 5. Algebraically, we are looking for a value x that is equal to $90 - x$. If we add x to both of these equal quantities, we get $2x = 90$. The only solution is $x = 45$, and the angle is **45** degrees.

Problem 6. From the second equation, we can determine that $a = b - 4$. Substituting this expression in place of a in the first equation, we get $b - 4 + 2b = 11$. This leads to $3b = 15$ and then $b = 5$. Knowing the value of b, we find that $a = 5 - 4 = 1$. Now we can determine the value of $4a - b$, which is $4 \times 1 - 5 =$ **−1**.

Problem 7. In base 3, the place values are powers of 3 and in base 2 the place values are powers of 2. The base-10 value of 1011_3 is $1 \times 27 + 0 \times 9 + 1 \times 3 + 1 \times 1 = 31$. Now we need to express the value 31 as a sum of powers of 2, which is $1 \times 16 + 1 \times 8 + 1 \times 4 + 1 \times 2 + 1 \times 1$. The base-2 numeral is thus $\mathbf{11111_2}$.

Problem 8. For each of the three odd digits (1, 3 or 5) that could be the units digit, there are four possible digits for the tens place and three possible digits for the hundreds place. Thus, there are $3 \times 4 \times 3 =$ **36** positive, odd integers that can be made.

Problem 9. Switching the coordinates of all the points has no effect on the area of the triangle, so the percent change is zero (**0**). The triangle is merely flipped over the line $y = x$.

Problem 10. Each of the eight students collected seven phone numbers, so cell phone numbers were entered $8 \times 7 =$ **56** times.

Warm-Up 8

Problem 1. If we buy both of the sets described, we will have 10 pencils and 10 pens for a cost of \$4.30 + \$5.20 = \$9.50. Five pencils and five pens would cost half that much, which is \$**4.75**.

Problem 2. All the numbers in this arithmetic sequence are 2 more than a multiple of 5. Palindromes read the same forward and backward. If we want our five-digit palindrome to be as small as possible, we had better have the first and last digits be 2s rather than 7s. The three middle digits can be zeros, so our number is **20,002**.

Problem 3. The bottom left and right triangles are each right triangles with legs of length 12 yards and 6 yards, so each triangle has an area of $(1/2) \times 12 \times 6 = 36$ square yards. The top white triangle is a 45-45-90 triangle with a hypotenuse of 12 and, therefore, legs of $6\sqrt{2}$. Its area is $(1/2)(6\sqrt{2})(6\sqrt{2}) = 36$ square yards. The square has an area of $12 \times 12 = 144$ square yards, so the area of the chevron is $144 - 3(36) =$ **36** square yards.

Problem 4. The length of the segment in question is $3/4 - 1/8 = 6/8 - 1/8 = 5/8$ units. If we trisect this segment, we are cutting it into three equal parts. We don't need to know what the trisection coordinates are to know that their difference is one-third of 5/8, or **5/24**.

Problem 5. Since the riding lawn mower mows five times as fast as the push mower, Manny can do 5 parts of the lawn while Timmy does 1 part of the lawn. That's 6 parts in all. Since Manny does 5/6 of the lawn, it should take $5/6 \times 90$ minutes, which is **75** minutes.

Problem 6. The units digits of the powers of 3 cycle through four digits in the following order: 3, 9, 7, 1. Since 75 has a remainder of 3 when divided by 4, the units digit of 3^{75} will be the third in the cycle, which is 7. Similarly, the units digits of the powers of 2 cycle through 2, 4, 8, 6. Since 113 has a remainder of 1 when divided by 4, the units digit of 2^{113} will be the first in the cycle, which is 2. The units digit of the product $(3^{75})(2^{113})$ will be the units digit of the product 7×2, which is **4**.

Problem 7. If we use 60 as a common denominator, the expression in the numerator simplifies to $(20 + 15 + 12)/60 = 47/60$ and the expression in the denominator simplifies to $(30 + 12 + 10)/60 = 52/60$. The ratio of these two values is simply **47/52**.

Problem 8. The prime factorization of 2010 is $2 \times 3 \times 5 \times 67$. To get the maximum sum of three different positive integers with a product of 2010, we should make two of the numbers as small as possible, namely 1 and 2. The third number would have to be $3 \times 5 \times 67 = 1005$. Thus, the sum is $1 + 2 + 1005 =$ **1008**.

Problem 9. Clearly we can use three of the boxes with a volume of 81 cubic meters. That leaves $300 - 3 \times 81 = 57$ cubic meters of sand. We can now use two boxes with a volume of 27 cubic meters and one box with a volume of 3 cubic meters, since $2 \times 27 + 3 = 57$. That's **6** boxes in all.

Problem 10. The function notation describes a general rule for turning inputs into outputs. The value of $F(4)$ is $3 \times 4 - 5 = 12 - 5 = 7$. The value of $G(-3)$ is $(-3)^2 + 3(-3) - 2 = 9 - 9 - 2 = -2$. The value of $H(-10)$ is $15 - 0.3 \times (-10) = 15 - (-3) = 18$. Thus, the value of the whole expression is $(7 \times 18)/(-2) = \mathbf{-63}$.

Workout 4

Problem 1. The volume of the ice formed will be $979 \times 12/11 = 89 \times 12 =$ **1068** cubic centimeters.

Problem 2. The product of the two legs of the right triangle would be twice its area, or $84 \times 2 = 168$. We need to look among the factors of 168 for two integers that are the legs of a Pythagorean triple. The prime factorization of 168 is $2^3 \times 3 \times 7$. It has 16 factors, as shown in the factor array at right, so there are 8 pairs to consider. Only the pair 7 and 24 leads to an integer length for the hypotenuse as shown below.

1	3	7	21
2	6	14	42
4	12	28	84
8	24	56	168

$h = \sqrt{7^2 + 24^2} = \sqrt{49 + 576} = \sqrt{625} = 25$

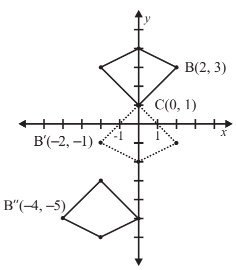
The perimeter of the triangle is, thus, $7 + 24 + 25 =$ **56** meters.

Problem 3. The least multiple of 3 that is 1 more than a multiple of 4 is 9. If we keep adding 12 to 9, we will preserve the criteria for 3 and 4 and we can look for a number that is 3 more than a multiple of 7. We get the arithmetic sequence 9, 21, 33, 45, and so on. Thus, **45** is our number.

Problem 4. Let's start by using 8 in place of u and 10 in place of z. Now we have the following: $8 + w = x$; $w + x = y$; $x + y = 10$; and $8 + w + x + y + 10 = ?$. Since each of the first three equations has an x, we can solve the first and third for x and substitute into the second equation: $(x - 8) + x = 10 - x \rightarrow 2x - 8 = 10 - x \rightarrow 3x = 18 \rightarrow x = 6$. Now our fourth equation can become $8 + (-2) + 6 + 4 + 10 =$ **26**.

Problem 5. The prime factorization of 245 is 5×7^2. We need a second factor of 5, so n must equal **5**.

Problem 6. The end result of the rotation and translation is shown. The coordinates of the final image of point B(2, 3) are B″**(–4, –5)**.

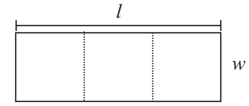


Problem 7. The price of gas rose by \$3.75 – \$0.40 = \$3.35 per gallon. That is an increase of $(3.35/0.4)(100) =$ **837.5**%.

Problem 8. There must be a 2 in one section and a 6 in one section. Neither is a factor of 27, so the other fours sections must contain the factors of 27. The factors of 27 are 1, 3, 9 and 27. Clearly the 27 is not used, since the sum of the odd numbers on the spinner is 16. Those odd numbers must be 1, 3, 3 and 9. The product of the six numbers on the spinner is $2 \times 6 \times 1 \times 3 \times 3 \times 9 = 12 \times 81 =$ **972**.

Problem 9. There are 8 single-region triangles and 8 double-region triangles, or **16** triangles in all.

Problem 10. Since the length of the rectangle is 3/8 of the perimeter, the sum of the two lengths in the perimeter is 6/8 of the perimeter. Each of the two widths of the rectangle must be 1/8 of the perimeter. This means the length of the rectangle must be 3 times the width. We can imagine cutting the rectangle into 3 congruent squares, each with an area of $18.75 \div 3 = 6.25$ square inches. The side of these squares must be $\sqrt{6.25} = 2.5$ inches. The length of the rectangle is 3 times this amount, or **7.5** inches.



Warm-Up 9

Problem 1. Since surface area is two-dimensional, the scale factor of the two similar objects is the square root of the ratio of their surface areas, which in this case is $\sqrt{(240/60)} = \sqrt{4} = 2$. With a scale factor of 2, the volume of the larger object is $2^3 = 8$ times that of the smaller object. The smaller object must have a volume of $216 \div 8 =$ **27** cubic centimeters.

Problem 2. We should disregard the fact that there are 5000 possible numbers. Whatever the numbers of the top three finishers are, we can be sure that there is a High, a Medium and a Low number. There are six ways to arrange the letters *H*, *M* and *L*. Only one of these arrangements is decreasing (*HML*), so the probability is **1/6**.

Problem 3. After Tricia gave away one-third of the marbles to Barbara, there were two-thirds left. She then gave one-fourth of the two-thirds that were left to Sam, which is $1/4 \times 2/3 = 1/6$ of the original marbles. At this point, she had given away exactly half the marbles, since $1/3 + 1/6 = 1/2$. The 24 marbles in the bag are the other half, so she must have had 48 to begin with. She gave Barbara $1/3 \times 48 =$ **16** marbles.

Problem 4. The surface area of the cube is $6 \times 50^2 = 6 \times 2500 = 15{,}000$. The volume of the cube is $50^3 = 125{,}000$. Thus, the difference between these numerical values is $125{,}000 - 15{,}000 =$ **110,000**.

Problem 5. Reflecting over the line $y = x$ switches the x and y coordinates, so P′ is (4, 1). The distance between (1, 4) and (4, 1) can be found with the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, which is based on the Pythagorean Theorem. We calculate as follows:

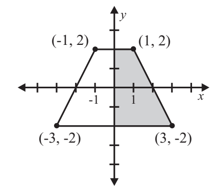
$d = \sqrt{(4-1)^2 + (1-4)^2} = \sqrt{3^2 + 3^2} = \sqrt{9 \times 2} = \mathbf{3\sqrt{2}}$ units.

Problem 6. The 12 two-digit numbers are <u>23</u>, 24, 27, 32, 34, <u>37</u>, 42, <u>43</u>, <u>47</u>, 72, <u>73</u> and 74, with the primes underlined. The ratio of primes to composites is 5 to 7 or **5/7**.

Problem 7. The total money raised was 1600 + 1500 + 900 + 2000 = \$6000. The seventh grade raised 900/6000 = 3/20 of the total. Three-twentieths of 360 degrees is 3/20 × 360 = **54** degrees.

Problem 8. There are 7! = 5040 ways to arrange the seven letters in ALABAMA, but since all the As are the same, we divide by the 4! = 24 ways that the four As can be arranged. That gives us 5040/24 = 210 distinct arrangements of the letters in ALABAMA. If we treat the four As as a single block of letters so they stick together, then there are 4! = 24 ways to arrange the block of As and the three other letters. The probability is thus 24/210 = **4/35** that the four As are together.

Problem 9. The full trapezoid has an area of 1/2 × 4 × (2 + 6) = 16. Exactly half of the trapezoid is to the right of the *y*-axis, where $x > 0$, and half of the trapezoid is to the left of the *y*-axis, where $x < 0$. (The *y*-axis itself has no breadth, so it contains no area.) Thus, the desired area is **8** square units.



Problem 10. We can factor out an x from the left side of the equation to get $x(64 + 16x - 4x^2 - x^3) = 0$, so we know that one of the roots is zero. Thus, the product of the roots is **0**. It is, however, possible to factor the expression and find the roots, as shown below. The roots are 0, 4 and a double root of −4.

$$64x + 16x^2 - 4x^3 - x^4 = 0$$
$$16x(4+x) - x^3(4+x) = 0$$
$$(16x - x^3)(4+x) = 0$$
$$x(16 - x^2)(4+x) = 0$$
$$x(4-x)(4+x)(4+x) = 0$$

Warm-Up 10

Problem 1. Let Josh mow the lawn himself in x minutes. Then we know that in 1 minute, Kevin would mow 1/40 of the lawn and Josh would mow $1/x$ of the lawn. Together those must be 1/15 of the lawn, so $1/40 + 1/x = 1/15 \rightarrow 3x + 120 = 8x \rightarrow 120 = 5x \rightarrow x =$ **24** minutes.

Problem 2. Since C + B = 1 in the units place and B + B = 1 in the tens place, there must be a 1 that is carried. This means that C is 1 more than B, and they can only be 6 and 5, respectively. Once we fill in all the 5s for B, we find that A = 4 and D = 1. Thus, the desired sum is 4 + 5 + 6 + 1 = **16**.

Problem 3. In each of the 9 sets of 100 three-digit integers (100s, 200s, and so on) there are 10 integers that end in 3, so there is a total of 10(9) = 90 three-digit integers that end with a 3. If there is a 3 in the units place already, then the sum of the hundreds place and the tens place must be a multiple of 3 for the whole number to be a multiple of 3. There are 30 such numbers, starting with 123, 153, 183, and ending with 903, 933, 963, 993. Thirty out of the 90 positive three-digit integers gives a probability of 30/90 = **1/3**.

Problem 4. Volume is maximized when the box is in the shape of a cube. There are 12 edges on a cube, so each edge would be 96 ÷ 12 = 8 inches long. The volume of the cube would be 8^3 = **512** cubic inches.

Problem 5. Since the drink is 10% juice, it is 90% non-juice. There are 4 quarts in 1 gallon. Ninety percent of one gallon is 0.9 × 4 = 3.6 quarts. Since the new mixture is to be 60% juice, these 3.6 quarts of non-juice will be the other 40% of the mixture. To get from 40% to 100%, we can multiply by 2.5, so the new mixture will be a total of 3.6 quarts × 2.5 = 9 quarts. This means that 9 − 4 = **5** quarts must be added.

Problem 6. The prime factorization of 1800 is $2^3 \times 3^2 \times 5^2$. Raising each exponent by one and multiplying the results gives us the total number of factors, which is 4 × 3 × 3 = 36 factors. The factors less than or equal to 15 are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12 and 15, which is 11 factors. There must be another 36 − 11 = **25** factors that are greater than 15. To confirm this, all 36 factors are given in the array below.

1	3	9	5	15	45	25	75	225
2	6	18	10	30	90	50	150	450
4	12	36	20	60	180	100	300	900
8	24	72	40	120	360	200	600	1800

Problem 7. There are 6 ways to get a sum of 5: 0 + 5, 1 + 4, 2 + 3, 3 + 2, 4 + 1 and 5 + 0. Half of the remaining 30 possible rolls give a sum below 5 and half give a sum above 5. Thus, there are 15 + 6 = 21 ways to roll a sum that is less than 6, which simplifies to a probability of **7/12**. Note that this question is the same as asking for the probability that the sum will be less than 8 when two standard dice are rolled.

Problem 8. Let x be the mean, the median and the range, which are all the same. We know that the smallest number is 17 and the largest number is $17 + x$. The median is the average of the second and third largest numbers and since the median is equal to x, the sum of the second and third largest numbers is $2x$. If we include another $2x$, then we will have the sum of all four numbers, since the mean is equal to x. We can solve for x as follows: $17 + (17 + x) + 2x = 4x \rightarrow 34 + 3x = 4x \rightarrow 34 = x$. The largest number must be 17 + 34 = **51**.

Problem 9. Since the center of the square is (3, 2) and one of the vertices is (5, 4), we can determine the other three vertices of the square are at (1, 4), (1, 0) and (5, 0). (We can do this using our knowledge of slope and the fact that the diagonals of a square are congruent, perpendicular bisectors of each other.) The side length of the square is 4 units. Thus, the area of the square is 4(4) = **16** square units.

Problem 10. Since the product of the three numbers in each row, column and diagonal is 1, $u = 1/(4 \times 1/8) = 2$, $r = 1/(2 \times 1) = 1/2$, $p = 1/(1 \times 1/8) = 8$ and $s = 1/(2 \times 8) = 1/16$. The value of $r + s$ is $1/2 + 1/16 =$ **9/16**.

Workout 5

Problem 1. Area is two-dimensional, so if we double all the dimensions of the room, then we will quadruple the surface area of the room since $2^2 = 4$. It will take the painter $2 \times 4 =$ **8** days.

Problem 2. The total miles traveled by the five vehicles is $5 \times 200 = 1000$, and the total number of gallons used is $14 + 12 + 12 + 6 + 8 = 52$. The average miles per gallon is $1000/52 = 250/13 =$ **19.2** mpg, to the nearest tenth.

Problem 3. The perimeter of an equilateral triangle is $3s$, and the area is $\sqrt{3}/4 \times s^2$, where s is the side length. Although perimeter and area are very different things, we can say that they are numerically equal and write the equation $3s = \sqrt{3}/4 \times s^2$. Since s clearly has a non-zero value, it is safe to divide both sides by s, which gives us $3 = \sqrt{3}/4 \times s$. Solving this for s, we get $s = 12/\sqrt{3}$, which is about **6.9** units, to the nearest tenth.

Problem 4. Since one book and five CDs cost half as much as three books and eight CDs, two books and 10 CDs would cost the same as three books and eight CDs. When this is written algebraically, with b equal to the price of one book and c equal to the price of one CD, we get $3b + 8c = 2b + 10c$, which reduces to $b = 2c$. We are told that one book and one CD cost \$45. If we trade the book for two CDs, we will have three CDs for \$45, which means CDs are \$15 each and books are \$30 each. The cost of six books is $6 \times \$30 =$ \$**180**.

Problem 5. Since the area of the largest triangle is 1, we know each leg of the isosceles triangle measures $\sqrt{2}$ units. The problem states that all of the triangles are similar, so the shaded triangles as well as the smaller unshaded triangles must be isosceles right triangles. In order for this to be true, the largest shaded triangle must have sides half as long as the sides of the largest triangle, which would be $\sqrt{2}/2$. Thus, the largest shaded triangle has an area of $(1/2)(\sqrt{2}/2)(\sqrt{2}/2) = 1/4$. Notice that the area of the largest shaded triangle is 1/4 of the area of the largest triangle. This relationship will continue as we work our way down the area measures of the smaller shaded triangles. Thus, the areas of the shaded triangles are 1/4, 1/16, 1/64 and 1/256. The sum of these areas is **85/256** square units.

Problem 6. The integers currently under the radical equal $2 + 4 + 6 = 12$ and, since $x < 8$, we can add a maximum of 7 to get a maximum total of $12 + 7 = 19$ under the radical. So let's look at perfect squares that are less than 19: 16, 9, 4, 1 and 0. Thus, the values of x that produce an integer value for the expression $\sqrt{(12 + x)}$ are $16 - 12 = 4$, $9 - 12 = -3$, $4 - 12 = -8$, $1 - 12 = -11$ and $0 - 12 = -12$. Their sum is $4 + (-3) + (-8) + (-11) + (-12) =$ **−30**.

Problem 7. The price of the original 10-carat diamond is $10^2 = 100$ times some constant k, and the sum of the prices of the 4-carat and 6-carat diamonds is $4^2 + 6^2 = 16 + 36 = 52$ times some constant k. The ratio of the sum of the smaller two to the larger one is $52k/100k =$ **52**%.

Problem 8. We can write the equation $40x^2 = 90$ and solve for x to find the rate of the annual increase in participants. $x = \sqrt{(90/40)} = \sqrt{(9/4)} = 3/2$, or 1.5. This means the increase in the number of participants was 50% each year. Since the study started with 40 participants, it must have added 20 participants by the end of the first year. Thus, the value of P is $40 + 20 =$ **60**.

Problem 9. In order to score 64 points, one dart will need to hit the center circular region and two darts will need to hit the outer ring. The area of the inner circle is π square units, and the area of the larger circle is 4π square units. Given that a dart has hit the target, the probability that it landed in the 50-point area is $\pi/4\pi = 1/4$, and the probability that it landed in the 7-point area is $3\pi/4\pi = 3/4$. The probability that the first dart lands in the center and the other two darts land in the outer ring is $1/4 \times 3/4 \times 3/4 = 9/64$. We should triple this probability since it could have been the second dart or the third dart that landed in the center. The probability that the total score is 64 points is thus $3 \times 9/64 =$ **27/64**.

Problem 10. The diagonal of the square is also the diameter of the circle. The radius of the circle is $7/2 = 3.5$ and its area is $\pi \times 3.5^2$ square units. The area of the square is half the square of its diagonal, which is $7^2 \div 2 = 24.5$ square units. The positive difference between these two areas is $(3.5^2\pi) - 24.5 =$ **13.98** square units, to the nearest hundredth.

Warm-Up 11

Problem 1. Shakeel made eight 2-point shots for 16 points. That leaves $30 - 16 = 14$ points. He could have made at most four 3-point shots for 12 points, so the minimum number of 1-point free throws he could have made is **2**.

Problem 2. The semi-perimeter of the rectangle is 31 meters. Since the length is 1 more than 4 times the width, the length plus the width is 5 widths and 1 more meter. The 5 widths must be 30 meters, so the width is 6 meters. That means the length is $4 \times 6 + 1 = 25$ meters. The area of the rectangle is $6 \times 25 =$ **150** square meters.

Problem 3. The coordinates of the midpoint are the averages of the x- and y-coordinates: $x = (5 + -7)/2 = -2/2 = -1$, and $y = (9 + -15)/2 = -6/2 = -3$. The midpoint is $(-1, -3)$, and the sum of the coordinates is **−4**.

Problem 4. If we multiply P/R by R/S, we will get P/S. The result is $2/3 \times 3/4 =$ **1/2**.

Problem 5. The triangle is the Pythagorean triple 6-8-10, so the hypotenuse is 10 units. The area of the triangle is $1/2 \times 6 \times 8 = 24$ square units. The sum of the areas of the two semicircles on the legs is equal to the area of the semicircle on the hypotenuse, so all three together make the area of a full circle with a diameter of 10 units. That would be $\pi \times 5^2 = 25\pi$ square units. Adding this to the area of the triangle, we get a total of **24 + 25π** square units.

Problem 6. There are $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways to arrange the five frogs. With respect to the colors, however, the three greens are interchangeable and the two yellows are interchangeable. Thus, there are only $120/(3! \times 2!) = 10$ distinct arrangements. If we imagine placing two greens on the ends, then there are only 3 distinct ways the frogs could be arranged: GGYYG, GYGYG and GYYGG. Therefore, the probability is **3/10** that there will be green frogs on both ends.

Problem 7. The expression $x^2 - 9$ is the difference of two squares, so $|x^2 - 9|$ can be factored into $|(x + 3)(x - 3)|$. The only way this product could be a prime number is if one of the factors is 1 or −1. This is the case when x is −4, −2, 2 or 4. The product of these is **64**.

Problem 8. Since there are 2 sets of 4 congruent angles, the sum of one angle from one set of congruent angles and one angle from the other set of congruent angles is 1/4 the total degrees in the internal angles of the octagon, or (1/4)(1080) = 270 degrees. Thus, B + C = $x + y + 50 = 270$ degrees. So, $x + y =$ **220** degrees.

Problem 9. Each small circle with half the radius of the larger circle has $(1/2)^2 = 1/4$ the area of the larger circle, so the two small circles together have the same area as the large semicircle. The difference is **0** square units.

Problem 10. There are "5 choose 3" ways the three common elements can be chosen. That's $(5 \times 4 \times 3)/(3 \times 2 \times 1) = 10$ ways. Now the remaining two elements can be either together in one of the sets or in two different sets, so that's $2 \times 10 =$ **20** ways the two subsets can meet these conditions.

Warm-Up 12

Problem 1. The original fraction is $(n - 8)/n$ and the fraction equal to 1 is $2(n - 8)/(n - 1)$. When the fraction is equal to 1, the numerator and denominator are equal, so $2(n - 8) = (n - 1)$. This means $2n - 16 = n - 1$, which leads to $n = 15$. The original fraction must have been $(15 - 8)/15 =$ **7/15**.

Problem 2. Since AP = 2PB, the segment AB is three times the length of AP. The horizontal distance between (−2, 5) and (10, 8) is 12 units. Point P must be two-thirds of this distance from point A, so its x-coordinate is $-2 + 8 = 6$. Similarly, it must be two-thirds of the vertical distance from point A, so its y-coordinate is 7. The coordinates of point P are **(6, 7)**.

Problem 3. The factor 5^2 is to be multiplied by a total of $5 + 11 + 17 + 78 + 14 = 125$, which is 5^3. Now, $5^2 \times 5^3 = 5^5$, so $x =$ **5**.

Problem 4. Factoring 783 into $3^3 \times 29$, we can see fairly quickly that 27 and 29 are a positive factor pair with two consecutive odd integers. Their product is indeed 783, and their sum is **56**.

Problem 5. It is convenient to choose 16 units as the length of the screen. Then the white rectangle has a width of 9 units and the larger rectangle has a width of $3 \times 4 = 12$ units. Since the lengths are the same, the ratio of the two areas is just the ratio of the two heights, which is 9/12, or 3/4. Thus, the white rectangle occupies **75%** of the largest rectangle. This, by the way, is what happens when you display the wide screen format on a standard television.

Problem 6. From the problem we get that $(a^2 + b^2 + c^2)/3 = 3((a + b + c)/3)^2$ and $b = 2a$. Through substitution we can eliminate the bs and just solve for the variable arrangement in question, c/a. The algebra for this problem is shown below. $c/a =$ **−2/3**.

$$\begin{aligned}
\frac{a^2+b^2+c^2}{3} &= 3\left(\frac{a+b+c}{3}\right)^2 \\
\frac{a^2+(2a)^2+c^2}{3} &= 3\left(\frac{a+2a+c}{3}\right)^2 \\
5a^2+c^2 &= 9\times\frac{(3a+c)^2}{9} \\
5a^2+c^2 &= 9a^2+6ac+c^2 \\
-4a^2 &= 6ac \\
-2a &= 3c \\
\frac{-2}{3} &= \frac{c}{a}
\end{aligned}$$

Problem 7. The fractal design at Stage 1 is (4)(1/2) = 4/2 = 2 units long. From stage to stage, the number of segments is multiplied by 4 while the segment lengths are cut in half. This means the total length of the segments doubles at each stage. Stage 2 is 4 units long, Stage 3 is 8 units long and Stage 4 is 16 units long. Thus, Stage 4 is $16 - 2 =$ **14** units longer that Stage 1.

Problem 8. There are three different types of squares to consider: corner squares, side squares and center squares. There is a 4/16 = 1/4 chance that a corner square is chosen first, for each of which 12 of the 15 remaining squares will not have a vertex in common. There is an 8/16 = 1/2 chance that a side square is chosen first, for each of which 10 of the 15 remaining squares will not have a vertex in common. There is a 4/16 = 1/4 chance that a center square is chosen first, for each of which 7 of the remaining 15 squares will not have a vertex in common. Thus, the probability of two squares not sharing a vertex is (1/4)(12/15) + (1/2)(10/15) + (1/4)(7/15) = (12/60) + (10/30) + (7/60) = 39/60 = **13/20**.

Problem 9. At the points where these curves cross, the y-values are certainly equal, so we can set the two equations equal to each other and solve for x: $x^2 - 8x - 35 = -2x^2 + 16x + 3 \rightarrow 3x^2 - 24x - 38 = 0$. This trinomial does not factor nicely into the product of two binomials. We could use the quadratic formula, but we don't actually need to find the two roots. Their sum is given in the trinomial; it's the opposite of the coefficient of the x term divided by the coefficient of the x^2 term. The answer is $-(-24)/3 =$ **8**.

Problem 10. The measure of angle BAC is one-half the measure of arc BC, so it's 30 degrees. The fact that segments AC and BE are perpendicular tells us that triangle ABD is a 30-60-90 triangle and that segment BE is bisected by segment AC. If we connect points A and E with a segment, then we would create equilateral triangle ABE. The length of segment AB is the same as segment BE, so it's **12** units.

Workout 6

Problem 1. If we combine all three statements, we get 8 apples, 8 bananas and 8 carrots for a total of \$5.92. Dividing by 8, we find that one of each fruit costs **74** cents.

Problem 2. The total score for the girls was 12 × 92 = 1104, and the total score for the boys was 10 × 87 = 870. The total points for the whole class was 1104 + 870 = 1974, so the mean test score for the whole class was 1974 ÷ 22 = **89.73** points, to the nearest hundredth.

Problem 3. The original price of the car was \$11,500 + \$4500 = \$16,000. Tria paid 11,500 ÷ 16,000 × 100 = 71.875%, or **71.9**% to the nearest tenth.

Problem 4. Let t be the time spent riding the bike. Since the walking speed of 3 miles per hour is 3/8 times as fast as the riding speed of 8 miles per hour, Scotty will spend 8/3 the amount of time walking as he does riding on the bike. In all, the time riding plus the time walking is $(11/3)t$. Since the total time is 11 hours, $(11/3)t = 11$, which means that $t = 11 \times 3/11 = 3$ hours. The person can ride for 3 hours at 8 miles per hour for a distance of 3 × 8 = **24** miles. He can then walk for 8 hours at 3 miles per hour to come back the same distance.

Problem 5. The volume of the packing box is 22 × 18 × 6 = 2376 cubic inches. The combined volume of the DVD player and the box for the power cord is (15 × 12 × 2) + (3 × 3 × 8) = 360 + 72 = 432. The difference 2376 − 432 = **1944** cubic inches is the volume of packing material Jeremy must use.

Problem 6. We can solve this problem by repeatedly changing the price of gismos and calculating the expected revenue as shown at the right. The maximum revenue occurs when she sells 20 gismos at \$40 each for \$**800** in revenue. We could also write the following quadratic equation, where x is the number of times Stella raises the price and y is the amount of revenue: $y = (30 - 2x)(20 + 4x) = -8x^2 + 80x + 600$. The line of symmetry is at $-b/2a$, which, in this case, is $-80/-16 = 5$. When $x = 5$, $y = (30 - 2 \times 5)(20 + 4 \times 5) = 20 \times 40 =$ **800**.

Number ×	Price =	Revenue
30	\$20	\$600
28	\$24	\$672
26	\$28	\$728
24	\$32	\$768
22	\$36	\$792
20	**\$40**	**\$800**
18	\$44	\$792
16	\$48	\$768
14	\$52	\$728
12	\$56	\$672
10	\$60	\$600

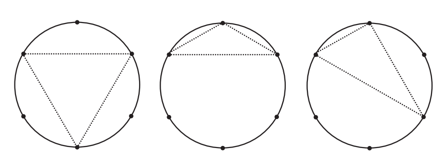
Problem 7. The area of the trapezoid is 1/2 × 6 × (12 + 24) = 108 square units. The midsegment EF is (12 + 24) ÷ 2 = 18 units, which is the average of the bases. The midsegment EF is also perpendicular to height line HG, which is 6 units. These two segments subdivide parallelogram HFGE into four congruent right triangles with legs of 9 and 3 units. The area of HFGE is, thus, 4 × 1/2 × 9 × 3 = 54, so the difference between the areas of trapezoid ABCD and parallelogram HFGE is 108 − 54 = **54** square units, which is exactly half of the trapezoid.

Problem 8. Fifteen feet down the length of the 60-foot ramp is one quarter of the way down, with three quarters of the ramp left to go. Assuming the ramp has a constant slope, we can determine that the height at this point is 3/4 × 24 = 18 feet, so the vertical support beam will need to be **18** feet long.

Problem 9. With 22/7 as an approximation for pi, as instructed by the problem, the area of the circle is $(22/7) \times 7^2 = 154$ square feet. The area of the rectangular patio is 20 × 14 = 280 square feet, so the area of the patio outside the circle must be 280 − 154 = 126 square feet. This represents 126 ÷ 280 × 100 = **45**% of the patio.

Problem 10. In all, there are "10 choose 6," which is (10 × 9 × 8 × 7 × 6 × 5)/(6 × 5 × 4 × 3 × 2 × 1) = 210 different ways to choose six marbles from the bag. The probability that two marbles of each color are drawn is the same as the probability that 3 reds and 1 blue remain in the bag. This is slightly easier to think about. There are "5 choose 3," which is (5 × 4 × 3)/(3 × 2 × 1) = 10 ways to choose the three reds, and "3 choose 1" which is 3 ways to choose the one blue. That's 10 × 3 = 30 ways to have three red marbles and one blue marble remain in the bag. The probability is thus 30/210 = **1/7**.

Warm-Up 13



Problem 1. Only three different triangles are possible, but they are not equally likely. There are "6 choose 3" ways to select three of the six points. That's (6 × 5 × 4) ÷ (3 × 2 × 1) = 20 sets of three points. Only 2 of these sets form equilateral triangles, and 6 of them form isosceles triangles (that are not equilateral). The other 12 sets of three points must form scalene triangles like the one on the far right or its mirror image. Thus, the probability is 12/20, or **3/5**.

Problem 2. The perpendicular bisector of the segment connecting the points (1, 3) and (7, 5) contains all points that are equidistant from the given points. If we draw the perpendicular bisector carefully on graph paper from the midpoint (4, 4) and with a slope of −3, we will find that it crosses the y-axis at (0, 16). We can also determine this point algebraically as follows. First we find the slope of the line segment that passes through (1, 3) and (7, 5). That slope is (5 − 3)/(7 − 1) = 1/3. The slope of the perpendicular bisector is the negative reciprocal of 1/3, which is −3. The perpendicular bisector passes through the midpoint of the segment, which is (4, 4). We now have the m and a particular x and y in the $y = mx + b$ form of the equation of the perpendicular bisector. We can solve the equation $4 = -3 \times 4 + b$ for b, and we find that $b =$ **16**, which is the y-coordinate we wanted.

Problem 3. The difference between the rate of flow of the water coming into the tank and that of the water leaving the tank is $75 - 24 = 51$ gallons per minute. It will take $600 \div 51 = 11.76$, or about 12 minutes to fill the 600-gallon tank. The time will be **7:42** am.

Problem 4. Given that the set of 5 numbers has a mean of 11, we can be certain that the sum of the 5 numbers is $5 \times 11 = 55$. One of the numbers must be 10 (the median), and at least two must be 7 (the mode). That means the remaining two numbers must add up to $55 - 10 - 2 \times 7 = 31$. Since we already have two 7s below the median, we have to find two numbers above the median so that the range is as great as possible. If we make one number 11, then the other can be 20. The range is $20 - 7 =$ **13**.

Problem 5. The jar contains $5 \times 4 = 20$ quarters, $5 \times 10 = 50$ dimes, $5 \times 20 = 100$ nickels and $5 \times 100 = 500$ pennies, which is $20 + 50 + 100 + 500 = 670$ coins in all. The probability that Connie will randomly choose a dime is $50/670 =$ **5/67**.

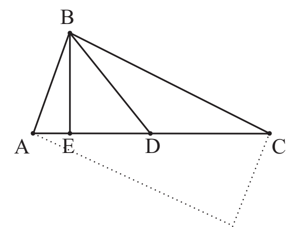
Problem 6. The formula for the volume of a cylinder is $V = \pi r^2 h$, where r is the radius (not the diameter) and h is the height. The ratio of the volume of the smaller cylinder to the larger cylinder is $(\pi \times 3^2 \times 20)/(\pi \times 4^2 \times 17)$, which reduces to 180/272 or **45/68**.

Problem 7. Starting with $-5 \le 2x - 7 \le 21$, we can add 7 to each of the expressions in the inequality to get $-5 + 7 \le 2x - 7 + 7 \le 21 + 7 \rightarrow 2 \le 2x \le 28$. Now we can divide each expression by 2 to get $1 \le x \le 14$. Thus, there are **14** integer values for x.

Problem 8. The three sums of two numbers on opposite sides of the cube are $3 + 5 = 8$, $2 + 7 = 9$ and, the largest, $4 + 6 =$ **10**.

Problem 9. Let's say that Bob drove at a rate of r miles per hour for t hours and covered a distance of $d = rt$ miles. Then Alice drove at a rate of $r + 5$ miles per hour for $t + 1$ hours and covered a distance of $(r + 5)(t + 1) = rt + 5t + r + 5$ miles. Since we are told that Alice drove 50 miles more than Bob, we can subtract Bob's distance from Alice's and find that $5t + r + 5 = 50$. Clark drove at a rate of $r + 10$ miles per hour for $t + 2$ hours and covered a distance of $(r + 10)(t + 2) = rt + 10t + 2r + 20$. This is $10t + 2r + 20$ miles more than Bob drove, which is equal to $2(5t + r + 5) + 10 = 100 + 10 =$ **110** miles.

Problem 10. In the figure shown, segment BD is the median (15 units) and segment BE is the altitude (12 units). Triangle BED is a multiple of the 3-4-5 Pythagorean triple, so the length of segment ED is 9 units. The median to the hypotenuse of a right triangle is equal to the two equal pieces it divides the hypotenuse into, so BD = DC = AD = 15 units. (If you consider triangle ABC as half of a rectangle, BD is half the rectangle's diagonal and segment AC is the other diagonal. Since diagonals are congruent and bisect each other in rectangles, BD = DC = AD.) Therefore, AE = 15 – 9 = 6 units. Using triangle AEB, we can use the Pythagorean Theorem to find the length of segment AB as follows: $6^2 + 12^2 = c^2 \rightarrow 36 + 144 = c^2 \rightarrow 180 = c^2 \rightarrow c = \sqrt{180} \rightarrow c = \sqrt{36} \times \sqrt{5} \rightarrow c = \mathbf{6\sqrt{5}}$ units.



Warm-Up 14

Problem 1. A subtraction problem is an addition problem in reverse in any base. If we work the problem backward, we get $154 + 123 = 321$. We can tell from the ones place that we must be in base **6**, since $4 + 3$ is one group of 6 and 1 left over. The base could not be 2 or 3, since the digits 3, 4 and 5 are used.

Problem 2. The left term $4(3^{n+2})$ can be rewritten as $4(3^n \times 3^2) = 36(3^n)$. Combining this with $45(3^n)$, we get $81(3^n)$, which equals 3. Since 81 is 3^4, n needs to be **−3** so that $3^4 \times 3^{-3} = 3^{4-3} = 3^1 = 3$.

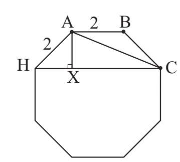
Problem 3. Triangles ADP and BCP are similar. We know that AD = 4 cm and BC = 8 cm, so since sides AD and BC are corresponding sides of similar triangles, the other sides of the triangle are going to relate in a 1:2 ratio as well. That means that DP/DC = 1/3 = DP/20, so DP measures 20/3. Now we can use the Pythagorean Theorem to solve for side AP. $(4)^2 + (AP)^2 = (20/3)^2 \rightarrow \sqrt{400/9 - 16} = AP$. $\sqrt{400/9 - 144/9} = \sqrt{256/9} =$ **16/3** cm.

Problem 4. The 5-by-6 rectangle could be placed in either "portrait" or "landscape" orientation, and the sum of the coordinates of the upper right vertex would be $6 + 9 =$ **15** or $7 + 8 = 15$.

Problem 5. The cross product gives us $2 \times 3^7 = N$, so $N = 4374$ and the units digit is **4**. Without multiplying out 3^7, we know the units digits cycle through 3, 9, 7, 1 for powers of 3. So, 3^7 has a units digit of 7 and when 7 is multiplied by 2, the units digit of the result will become 4.

Problem 6. If we put 2 wheels on all 67 cycles in the shop, we would use only $2 \times 67 = 134$ wheels. The extra $157 - 134 = 23$ wheels must belong to tricycles, so there are **23** tricycles in the shop.

Problem 7. Since the obtuse angle BAH is 135°, we can see that ABCH is an isosceles trapezoid with angles of 135° and 45°. Dropping altitude AX creates 45-45-90 triangle AXH, and we can determine that since AH = 2, AX = $\sqrt{2}$. If we use segment AB as the base of triangle ABC, then segment AX is its height and the area is $1/2 \times 2 \times \sqrt{2} = \mathbf{\sqrt{2}}$ square centimeters.



Problem 8. Three hundred is 1/4 of 1200 calories, so one must run for 15 minutes at 12 mph. Similarly, 300 is 2/3 of 450, so one must run 40 minutes at 5 mph. The desired difference is $40 - 15 =$ **25** minutes.

Problem 9. None of the female students at Houston High School will be moved, so the current 30% of 2000 students, which is 600 female students, will remain the same. If the school is to be 60% male instead of 70% male, then the same 600 female students must represent the other 40%. In that case, there would need to be a total of $600 \div 0.40 = 1500$ students in all. That means $2000 - 1500 =$ **500** students must be moved to another school.

Problem 10. Evens and odds are equally likely when rolling dice. There are $2^6 = 64$ different ways that six dice can land even or odd. The sixth row of Pascal's triangle (1, 6, 15, 20, 15, 6, 1) shows us the number of ways that all six can be even, then five of the six, then four of the six, and so on. Given that at least three of the men have rolled even numbers, then we are dealing with just 1 + 6 + 15 + 20 = 42 different possible situations. If we select John, Fred and Martin as having each rolled an even number, then the other three men might have rolled any of 2^3 = 8 different collections of evens and odds. Thus, the probability is 8/42, which reduces to **4/21**. To confirm this, we can consider the first four numbers on the sixth row of Pascal's triangle. Of the 1 way that all six men can roll evens, there is 1 way that John, Fred and Martin can roll evens. Of the 6 ways that just five men can roll evens, there are 3 ways that John, Fred and Martin can roll evens. Of the 15 ways that four men can roll evens, there are 3 ways that John, Fred and Martin can roll evens. Of the 20 ways that three of the men can roll evens, there is only 1 way that John, Fred and Martin can roll evens. Once again, we get 1 + 3 + 3 + 1 = 8 ways out of the 42 considered, which is 4/21.

Workout 7

Problem 1. Since we are looking for the least number of students that could participate in all three sports, let's start by seeing if 0 is a possibility. When we do this, we see that there are 29 students that participate in basketball, but there are 18 that play basketball and football, and there are 13 that play soccer and basketball. Since 18 + 13 = 31 and 31 − 29 = 2, at least **2** students must play all three sports.

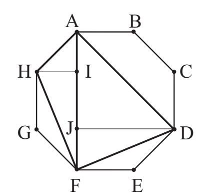
Problem 2. The triangular numbers are 1, 3, 6, 10, 15, and so on, and the square numbers are 1, 4, 9, 16, 25, and so on. There are many pairs of triangular numbers that add up to a square number, such as 3 + 6 = 9, 1 + 15 = 16, 21 + 28 = 49, and so on. Since we want to find the greatest triangular number less than 100 in such a pair, we should definitely try 91, which is the 13th triangular number. Indeed, 91 can be paired with 78 to make 169, which is 13 × 13, so **91** is our answer.

Problem 3. The increase of 50% in gas mileage means the new car gets 3/2 times the old gas mileage. Since the new car gets 24 miles per gallon, the old car must have gotten 2/3 × 24 = 16 miles per gallon. In her old car, she would have used 480 ÷ 16 = 30 gallons of gas for the trip to visit relatives. In her new car, she will use only 480 ÷ 24 = 20 gallons of gas. That's a savings of **10** gallons.

Problem 4. *Distance* equals *rate* times *time*, so *time* equals *distance* divided by *rate*. Mr. Davidson spent 30/50 = 3/5 of an hour driving to work. He spent 30/20 = 3/2 hours driving back. That's 3/5 + 3/2 = 21/10 hours, or **2.1** hours.

Problem 5. If exactly one of the statements is true, then four of them must be false, which is exactly what statement **D** says.

Problem 6. Triangles ADF and AHF share base AF, so the ratio of their areas is the same as the ratio of their altitudes. If we choose the length of altitude HI as our unit, then the length of altitude JD is √2 + 1 units. This is because the side length of the octagon is √2 times the unit HI (notice that triangle AHI is a 45-45-90 triangle) and JD is like a side length plus HI. The desired ratio is (√2 + 1)/1, which is **2.41** to the nearest hundredth.

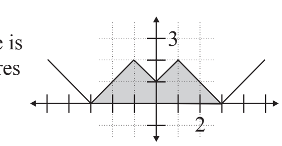


Problem 7. At the half-year mark, the balance on the investment will be \$1000 × 1.03 = \$1030. At the end of the year, the balance will be \$1030 × 1.03 = \$1060.90, which is \$**60.90** in interest. The extra 90¢ is due to interest on the interest that was posted at the half-year mark.

Problem 8. First let's count the squares. There are 4 squares made out of just two triangles, 4 squares made out of four triangles, 1 square made out of eight triangles, and 1 square made out of all 16 triangles. Now, the rectangles are all twice as long as they are wide. There are 4 rectangles made out of four triangles and 4 rectangles made out of eight triangles. That's 4 + 4 + 1 + 1 + 4 + 4 = **18** rectangles in all.

Problem 9. There would be 4! = 4 × 3 × 2 × 1 = 24 ways to arrange the four children if there were no restrictions. Now let's consider the arrangements that we do not want. If we imagine Brad-Cathy to be a unit, then there are 3! = 3 × 2 × 1 = 6 ways to arrange them with the other two children. Likewise, there are 6 ways if we imagine Cathy-Brad to be a unit. Subtracting these 12 ways from the 24, we get **12** ways that work.

Problem 10. From −3 to 3 on the *x*-axis, the graph of $y = \big|\,\big||x|-1\big|-2\big|$ makes the shape of the letter M. Since the slope is either 1 or −1 everywhere along the graph, it is easy enough to count the triangles (1/2 square unit each) and the squares (one square unit each). The total area is **7** square units.



Warm-Up 15

Problem 1. There are 16 × 16 = 256 ounces in 16 pounds. If all 76 of the Fantasy gum packages were 3-ounce packages, that would be a total of 76 × 3 = 228 ounces, which is 28 ounces less than 256 ounces. If we swap a 3-ounce package for a 10-ounce package, we get an increase of 7 ounces. Since 28 ÷ 7 = 4, and the total weight must be less than 16 pounds, we can do a maximum of three swaps, so that is **3** 10-ounce packages.

Problem 2. The ten possible sums that are less than 6 are as follows: 1 + 1 + 1, 1 + 1 + 2, 1 + 1 + 3, 1 + 2 + 1, 1 + 2 + 2, 1 + 3 + 1, 2 + 1 + 1, 2 + 1 + 2, 2 + 2 + 1 and 3 + 1 + 1. Only one of them has three 1s, so the probability is **1/10**.

Problem 3. If we rewrite the equation $x - 2y = 8$ in slope-intercept form, we get $y = (1/2)x - 4$. The line parallel to this one but through point P(4, 3) also has a slope of 1/2 but has a different *y*-intercept (usually called *b*). Using the specific solution (4, 3), we solve the equation $3 = 1/2 \times 4 + b$, which gives us $b = 1$. The equation is $\boldsymbol{y = (1/2)x + 1}$.

Problem 4. There are 3! = 3 × 2 × 1 = 6 ways to arrange the consonants H, N and K. For each of these ways, there are two different places we can insert the A. Thus, there are 6 × 2 = 12 ways to make words that start and end with a consonant. However, that includes HANK, so our answer is 12 – 1 = **11** ways.

Problem 5. Since the subcommittee must have more than 1 but fewer than 5 members, it must have 2, 3 or 4 members. "Eight choose 2" is $(8 \times 7) \div (2 \times 1) = 28$. "Eight choose 3" is $(8 \times 7 \times 6) \div (3 \times 2 \times 1) = 56$. And "8 choose 4" is $(8 \times 7 \times 6 \times 5) \div (4 \times 3 \times 2 \times 1) = 70$. In all, there are $28 + 56 + 70 =$ **154** subcommittees possible.

Problem 6. The perimeters of the hexagon and octagon must be multiples of 24, the least common multiple of 6 and 8. The individual perimeters could be 24, 48, 72 or 96 units. In each case, the sum of the two perimeters would be less than 200. There are **4** pairs of hexagons and octagons that can be made.

Problem 7. The nearest perfect square to 150 is 144, so $\sqrt{150}$ is more than 12. Similarly, $\sqrt{75}$ is less than 9. The integer closest to $\sqrt{150} + \sqrt{75}$ should be $12 + 9 =$ **21**.

Problem 8. The length of the side of a medium square is 10 units. Call the length of the side of a small square S units. The length of the bottom of the large square is equal to 5S units. The length of the right side of the large square is equal to 30 – S units. Setting the two expressions for the side length of the large square equal to each other gives S = 5. The sides of the large square are then 25 units, and its area is $25 \times 25 =$ **625** square units. The original rectangle is 30 units by 35 units.

Problem 9. The radius of the inscribed circle is also the height of an equilateral triangle inside the hexagon. Given that the height of the equilateral triangle is $4\sqrt{3}$, we can use the 30-60-90 relationships in the two triangles it forms to determine that the side length of the triangle must be 8 units. The area of such a triangle is $1/2 \times 8 \times 4\sqrt{3} = 16\sqrt{3}$. The hexagon is composed of 6 equilateral triangles, so its area is $6 \times 16\sqrt{3} =$ $\mathbf{96\sqrt{3}}$ square centimeters.

Problem 10. The angle adjacent to the given 130-degree angle has a measure of $180 - 130 = 50$ degrees. Opposite angles in a parallelogram are equal, so the angle that is x degrees is 50 degrees. The value of x is **50**.

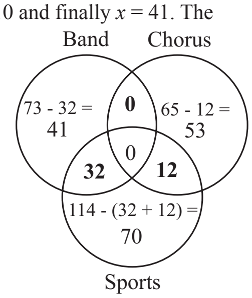
Warm-Up 16

Problem 1. In order to answer the question we must find the area of the circle and subtract it from the area of the parallelogram. Since the radius of the circle is 3, the area is $(3)^2\pi = 9\pi$ sq units. The problem told us that sides AB and CD are tangent to the circle; therefore, we know that the diameter of the circle is equal to the height of the parallelogram. Thus, the area of the parallelogram is $(10)(2 \times 3) = 60$. That means that the area within the parallelogram that is outside of the circle is $\mathbf{60 - 9\pi}$ sq units.

Problem 2. Since $f(x) = 5x$, we can substitute $5x$ in place of $f(x)$ in the statement of $g(x)$. Thus, $g(8) = 5 \times 8 - 3 \times 8 - 7 = 40 - 24 - 7 =$ **9**.

Problem 3. Since the sum of the digits is 8, the possible numbers are 17, 26, 35, 44, 53, 62 and 71. With some trial and error, we find that $71 = 3 + 4 \times 17$, so the original number was 17 and the new number is **71**. We can solve this algebraically if we call the digits a and b and consider the equation $10b + a = 3 + 4(10a + b)$. This simplifies to $6b = 3 + 39a$. Again, there is an opportunity to try small values of a and quickly find $a = 1$ and $b = 7$. But we can continue with the algebra by substituting $8 - a$ for b. This gives us $6(8 - a) = 3 + 39a$, which simplifies to $48 = 45a + 3$, so $a = 1$.

Problem 4. The sum of the angles in a quadrilateral is 360 degrees. Translating the English to algebra, we can use x for the smallest angle to write the following equation: $4x + (2x + 10) + (3x - 60) + x = 360$. This simplifies to $10x - 50 = 360$, then $10x = 410$ and finally $x = 41$. The largest angle is $4 \times 41 =$ **164** degrees.



Problem 5. Based on the problem, we know that 32 students participate in band AND sports, 12 student participate in chorus AND sports, and 0 students participate in band AND chorus. These numbers are in bold in the Venn diagram shown. Because we know that no students are in both the band and the chorus, we know that 0 students participate in all three activities. Now by subtracting the areas of overlap from the band, chorus and sports totals, we can find out how many students participate in only one activity. This is shown within each Venn diagram circle. Now if we add each of these numbers to the number of students that do not participate in any of the activities, we will know the total number of students: $41 + 0 + 53 + 0 + 32 + 12 + 70 + 558 =$ **766** students.

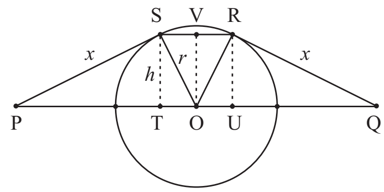
Problem 6. From any vertex of an n-gon (a polygon with n sides), $n - 3$ diagonals can be drawn. This is because no diagonal would be drawn to the vertex itself or to either of the adjacent vertices. If we multiply $(n - 3)$ by n, we get twice the number of diagonals since each diagonal has two ends. The formula for the number of diagonals in an n-gon is, thus, $d = (n - 3)n/2$. Given that we have a polygon with 20 diagonals, we can solve the equation $20 = (n - 3)n/2$ for n. This means that $40 = (n - 3)n \rightarrow n^2 - 3n - 40 = 0 \rightarrow (n - 8)(n + 5) = 0$, so $n = 8$ or $n = -5$. Thus, the polygon must have **8** sides.

Problem 7. First, notice that the denominator includes an x. Since the denominator cannot be 0, we know that x cannot be 2; thus, 2 is a point of interest on the number line. Now we can take the cross product and then solve the inequality as follows: $(x + 2)(x - 2) < 5 \rightarrow x^2 - 4 < 5 \rightarrow x^2 < 9 \rightarrow -3 < x < 3$. This tells us that 3 and –3 are points of interest on the number line as well. If we check a number between each of the interest intervals, we'll find that the $-10 \le x < -3$ range works and the $2 < x < 3$ range works. That means 8 of the 20 unit segments between -10 and 10 work, which is **40%**.

Problem 8. First we do the triangle operation inside the parentheses: $(5 - 3)/(2 \times 3 + 6) = 2/12 = 1/6$. Now we do the triangle operation again, with 3 and 1/6: $(3 - 1/6)/(2 \times 1/6 + 6) = (17/6)/(19/3) = 17/6 \times 3/19 =$ **17/38**.

Problem 9. Any integer with a digit of zero will give us a product of zero when we multiply the digits. There are 2 numbers that have three zeros: 3000 and 4000. There are three ways to have exactly two zeros: 300_, 30_0, and 3_00. For each of these ways, there are nine digits to place in the blank, so this gives us 3 × 9 = 27 numbers with exactly two zeros. There are three ways to have exactly one zero: 30_ _, 3_ _0 and 3_0_. For each of these ways, there are 9 × 9 = 81 ways to fill in the blanks, so this gives us 3 × 81 = 243 numbers with exactly one zero. This is a total of 2 + 27 + 243 = **272** integers.

Problem 10. If the circle is to be tangent to both segments PS and QR, the center of the circle must be at the midpoint of base PQ. The length x will be as small as possible when the tangent points are at S and R. We're going to use the Pythagorean Theorem on three different triangles, PST, OTS and OSP, and we will be dealing with three unknowns, x, h and r. Here's what we have: $x^2 + r^2 = 50^2$ (from ΔOSP), $13^2 + h^2 = r^2$ (from ΔOST) and $37^2 + h^2 = x^2$ (from ΔPST). Solving the second and third equations for h^2, we get $h^2 = r^2 - 13^2$ and $h^2 = x^2 - 37^2$. This means that $r^2 - 13^2 = x^2 - 37^2$, or $r^2 - 169 = x^2 - 1369$. Rearranging things a little, we get $x^2 - r^2 = 1200$. If we add this to the first equation, $x^2 + r^2 = 50^2$, we get $2x^2 = 3700$, so $x^2 = 3700/2 =$ **1850**, which is the smallest possible value for x^2.



Workout 8

Problem 1. There are 5280 feet in 1 mile, so 3 miles is 3 × 5280 = 15,840 feet, or 12 × 15,840 = 190,080 inches. Since Jonathan's tires revolved 2327 times, their circumference must be 190,080 ÷ 2327 ≈ 81.68 inches. Dividing this circumference by π, we get a diameter of **26** inches, to the nearest whole number.

Problem 2. The least common multiple of 1, 2, 3, 4, 5 and 6 is 60. This means the six friends will eat together every 60 days. They can get together **5** more times in that year, but not 6 times. Six times would be 360 days after January 15, which would be January 9th or 10th of the next year.

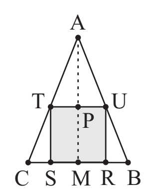
Problem 3. Seventy-five percent of 80% is 0.75 × 0.8 = 0.6, or 60% of the kettle. If 1200 ml is 60%, then 2000 ml, or **2** liters, would be 100%.

Problem 4. The radius of the circle is 6 inches, so the area of the circle is 36π square inches and the diameter is 12 inches. Since the square is inscribed, this length is the diagonal of the square. The area of a square equals $d^2/2$, where d is the length of the diagonal. Thus, the area of the square equals $12^2/2 = 72$ and the area inside the circle but outside the square is $36\pi - 72$, which is **41** square inches, to the nearest whole number.

Problem 5. Since 2000 ÷ 150 = 13 1/3, only 13 customers will get a free bag of groceries. That's 13/2000 × 100 = **0.65%** of the customers.

Problem 6. If we combine Tom's purchase and Sam's purchase, we get 3 shirts and 3 pairs of socks for \$15 + \$12 = \$27. That means 1 shirt and 1 pair of socks would cost \$9. Since Tom bought an extra shirt, we know that the shirts cost \$15 – \$9 = \$6 each. Since Sam bought an extra pair of socks, we know that the pairs of socks cost \$12 – 9 = \$3 each. Thus 15 shirts and 10 pairs of socks would cost 15 × \$6 + 10 × \$3 = \$90 + \$30 = \$**120**.

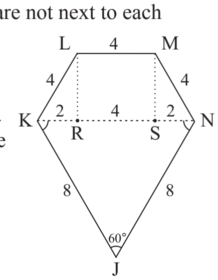
Problem 7. If we drop a perpendicular from point A, we split triangle ABC into two congruent right triangles with side lengths of 5, 12 and 13 feet. Triangle APU is similar to these 5-12-13 triangles. If we call the side length of the square $2x$, then the length PU is x and the length AP is $12 - 2x$. We can then set up the proportion $(12 - 2x)/12 = x/5$. The cross product gives us $60 - 10x = 12x$. This simplifies to $60 = 22x$, and then $x = 60/22$. The side length of the square is $2x$, so it's 2 × 60/22 = **60/11** feet.



Problem 8. Since the slope of the line QP is 1, we need to move 6 units to the left of P(4, 6) and we drop 6 units down from P(4, 6). The coordinates of Q are thus (−2, 0). The slope of line RP is 2, so we move only 3 units to the left of P(4, 6) as we come down 6 units. The coordinates of R are thus (1, 0). The base QR of triangle PQR is 3 units (from −2 to 1) and the height is 6 units. The area is 1/2 × 3 × 6 = **9** square units.

Problem 9. There are 6! = 6 × 5 × 4 × 3 × 2 × 1 = 720 ways that the six people can line up. If we treat Andy and Becky as a unit and have them move together, there are 5! = 120 ways to arrange the friends with Andy directly to the left of Becky and 5! = 120 ways to arrange the friends with Andy directly to the right of Becky. In the other 720 − (2 × 120) = 480 ways to arrange the friends, Andy and Becky are not next to each other. Thus, the probability is 480/720 = **2/3**.

Problem 10. Since $m\angle J = 60°$ and sides JK and JN are the same length, if we draw line KN, it will form equilateral triangle JKN, with KN also measuring 8 units. The area of this equilateral triangle would be $(1/2)(8)(4\sqrt{3}) = 16\sqrt{3}$ sq units. Notice that by drawing diagonal KN, we also form isosceles trapezoid KLMN. By dropping two lines perpendicular to line KN from L and M to points R and S, respectively, we see that right triangles are formed and we can use the Pythagorean Theorem. Thus lines LR and MS each measure $2\sqrt{3}$ units in length. Thus the area of trapezoid KLMN is $(1/2)(4 + 8)(2\sqrt{3}) = (1/2)(12)(2\sqrt{3}) = 12\sqrt{3}$ square units. That means that the area of JKLMN is $16\sqrt{3} + 12\sqrt{3} =$ $\mathbf{28\sqrt{3}}$ square units.



Warm-Up 17

Problem 1. If Sam turns in his paper on time with only 14 out 30 problems correct, he will get 10/10 × 14/30 = 140/300. If he turns it in 1 day late, he will get 9/10 × 18/30 = 162/300. If he hands it in 2 days late, he will get 8/10 × 22/30 = 176/300. At 3 days late, he will get 7/10 × 26/30 = 182/300. At 4 days late, he will get 6/10 × 30/30 = 180/300. Sam should turn in his paper **3** days late.

Problem 2. Since each box must contain at least 1 marble, the only question that remains is where to put the 3 extra marbles. There are 5 different ways to place all 3 in one box. There are $5 \times 4 = 20$ ways to place 2 in one box and 1 in another. There are "5 choose 3," or 10 ways, to place a single marble in three different boxes. That's $5 + 20 + 10 =$ **35** ways in all. Another solution is to see that there are 3 marbles left as shown: O O O. Since there are five boxes, we can think of this as four dividers (I). Some possible scenarios are IIOIOIO (a marble in the 3rd, 4th and 5th boxes), OOIIIOI (2 marbles in the 1st box and 1 in the fourth box) and IIIIOOO (3 marbles in the 5th box). There are seven slots and you need to select 4 for the dividers, so there are "7 choose 4" = 35 ways to put 3 marbles into five boxes.

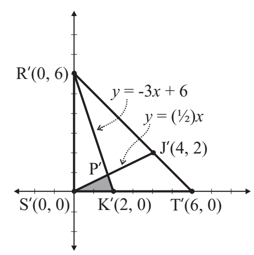
Problem 3. The product of the GCF and the LCM of two numbers is equal to the product of the two numbers. In this case, we have $3 \times 216 = 648$. The prime factorization of 648 is $2^3 \times 3^4$. The two numbers share a factor of 3 and cannot share a factor of 2 or any additional 3s, so they must be $2^3 \times 3$ and 3^3, which are 24 and 27. Their sum is **51**.

Problem 4. The x-values increase by 4 while the y-values increase by 3. This means the slope m is 3/4. Using this slope and one ordered pair from the table, say (1, 2), we can plug three numbers into the equation $y = mx + b$ and solve for the unknown b as follows: $2 = 3/4 \times 1 + b$, which means $b = 2 – 3/4 = 5/4$. The product of m and b is $3/4 \times 5/4 =$ **15/16**.

Problem 5. The area of triangle AKB is $(1/2) \times 4 \times 4 = 8$ square units. We will subtract from this the areas of triangles AJM and MLB. Triangles AJM and KJD are similar. Since AM = 2 and DK = 3, the ratio of the altitudes of AJM and KJD is 2:3; the sum of these altitudes is 4 units. Dividing 4 into 5 equal parts, we find that each part is 4/5 units. The altitude of AJM is two of these parts, or 8/5 units, so the area of triangle AJM is $(1/2) \times 2 \times 8/5 = 8/5$ square units. Triangles MLB and CLK are also similar. Since MB = 2 and KC = 1, the ratio of their altitudes is 2:1; the sum of their altitudes is 4 units. Dividing 4 into 3 equal parts, we find that each part is 4/3 units. The altitude of MLB is two of these parts, or 8/3, so the area of triangle MLB is $1/2 \times 2 \times 8/3 = 8/3$. Finally, the area of quadrilateral JKLM is $8 – 8/5 – 8/3 = (120 – 24 – 40)/15 =$ **56/15** square units.

Problem 6. If n were a prime, there would be a single pair of factors (divisors) with a product equal to n, namely $1 \times n$. If n were the product of two primes, there would be 2 pairs of factors, each with a product of n, so the product of all the factors would be n^2. We want the product of all the factors to be n^8, so we need 8 pairs of factors that multiply to n, or 16 factors in all. Recall that we can calculate the number of factors from the prime factorization of a number by raising each exponent in the prime factorization by one and then multiplying them. Working this in reverse, we can construct a number with 16 factors if we look at factors of 16, subtract one from each factor and then use these as exponents on the smallest primes. If we use $16 \times 1 = 16$, we get the number $2^{15} \times 1 = 32{,}768$, which has 16 factors. If we use $8 \times 2 = 16$, we get the number $2^7 \times 3 = 128 \times 3 = 384$, which has 16 factors. If we use $4 \times 4 = 16$, we get the number $2^3 \times 3^3 = 8 \times 27 = 216$, which has 16 factors and is smaller. If we use $4 \times 2 \times 2$, we get the number $2^3 \times 3 \times 5 = 8 \times 15 = 120$, which is smaller still. If we use $2 \times 2 \times 2 \times 2 = 16$, we get $2 \times 3 \times 5 \times 7 = 210$, which is bigger than 120. Thus, **120** is the smallest positive integer n whose factors have a product of n^8.

Problem 7. The area of ΔRPS is equal to the area of ΔRKS minus the area of ΔSPK. We know that the area of ΔSPK is 7, so if we can find the area of ΔRKS, we will know the area of ΔRPS. Since ΔRKS and ΔSPK have the same base (KS), the ratio of their areas is equal to the ratio of their heights. Although we are given the ratios for a specific triangle, they will hold for any triangle that has the sides divided into the same ratios, so we can consider right ΔR′S′T′ with R′ = (0, 6), S′ = (0, 0) and T′ = (6, 0). If we let J′ be the point on line segment R′T′ with R′J′:J′T′ = 2:1 and K′ be the point on S′T′ with T′K′:K′S′ = 2:1, we will have the same ratios as in the original ΔRST. The equation of the line segment S′J′ is $y = (1/2)x$ and the equation of the line segment K′R′ is $y = −3x + 6$. The two line segments intersect when $(1/2)x = −3x + 6$, which occurs when $x = 12/7$. Since $y = (1/2)x$, $y = 6/7$. Since the height of ΔR′S′T′ is 6, the ratio of the two heights is (6/7):6 = 1:7, so the height (and area) of ΔR′S′K′ is seven times the height (and area) of ΔS′P′K′. Thus, the area of ΔR′P′S′ is 7 – 1 = 6 times the area of ΔS′P′K′, which is $6 \times 7 =$ **42** sq units.



Problem 8. The minute hand moves $360 ÷ 60 = 6$ degrees per minute and the hour hand moves $360 ÷ 12 ÷ 60 = 0.5$ degree per minute, so the minute hand exceeds the hour hand by 5.5 degrees per minute. The hour hand and minute hand will next be perpendicular again when the minute hand has traveled 180 degrees more than the hour hand. Dividing 180 by 5.5, we get $180/5.5 = 1800/55 = 360/11 =$ **32 8/11** minutes.

Problem 9. If the 100 red balls are numbered from 1 to 100 from left to right, then the multiples of four are replaced by green balls. That leaves $3/4 \times 100 = 75$ balls that are still red. When we start from the right for the multiples of five, however, it is not the multiples of five that get replaced. The ball numbers are 96, 91, 86, 81, and so on, which are all one more than multiples of five. We replace $1/5 \times 100 = 20$ balls with the white balls, but 5 of those balls—numbers 96, 76, 56, 36 and 16—were green. Thus, we removed only 15 red balls, which leaves $75 − 15 = 60$ red balls at this point. Now, starting from the left, we replace the multiples of six with yellow balls. There are 16 multiples of six less than 100, but 8 of them—the even multiples of six—are also multiples of four that have already been replaced by a non-red ball. Among the 8 odd multiples of six, there are 2 balls that have been changed to white—ball 6 and ball 66. Thus only 6 red balls are removed for the multiples of six. That leaves $60 − 6 =$ **54** red balls.

Problem 10. A regular hexagon is made up of 6 equilateral triangles, each of which has a height of $\sqrt{3}/2$ times the side length s. The area of a regular hexagon is, therefore, $6 \times 1/2 \times s \times \sqrt{3}/2 \times s = (3\sqrt{3}/2)s^2$. In the figure, triangles AHD and BHC are similar, with AD = 2 × BC. If we use AD = $2s$ as the base of triangle AHD, then its height is 2/3 of the height of an equilateral triangle, so it's $2/3 \times \sqrt{3}/2 \times s$, or $(\sqrt{3}/3)s$. The area of AHD is, thus, $1/2 \times 2s \times (\sqrt{3}/3)s = (\sqrt{3}/3)s^2$. The desired ratio is $(\sqrt{3}/3)s^2/(3\sqrt{3}/2)s^2 = (\sqrt{3}/3)/(3\sqrt{3}/2) =$ **2/9**.

Warm-Up 18

Problem 1. The equation $x^2 + y^2 = 36 − 2xy$ can be rewritten as $x^2 + 2xy + y^2 = 36$ and then again as $(x + y)^2 = 36$. This means that $x + y = 6$ (or −6). The equation $x^2 − y^2 = 12$ can be rewritten as $(x + y)(x − y) = 12$. Since we know that $x + y = 6$ (or −6), it follows that $x − y = 2$ (or −2), since $6 \times 2 = 12$. Knowing the sum and the difference of x and y, we can quickly find that $x = 4$ and $y = 2$ (or $x = −4$ and $y = −2$). Their product is, therefore, **8**.

Problem 2. It is not necessary to find the particular sums involved here. The sum of all the units digits in any "decade" is 0 + 1 + 2 + … + 9 = 45. There are 9 of these decades among the two-digit numbers, so that's 9 × 45 = 405 so far. The sum of all the tens digits is 10 × 1 + 10 × 2 + 10 × 3 + … 10 × 9 = 10 × (1 + 2 + 3 + … + 9) = 10 × 45 = 450. Putting the units digits and the tens digits together, we get a sum of 405 + 450 = **855**.

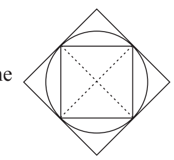
Problem 3. We will subtract the areas of the three unshaded triangles from $12^2 = 144$, which is the area of the square. Triangle PQT is a 30-60-90 triangle, so we know that the hypotenuse is twice the length of the shorter leg. That leads to the following use of the Pythagorean Theorem: $12^2 + x^2 = (2x)^2 \rightarrow 144 + x^2 = 4x^2 \rightarrow 144 = 3x^2 \rightarrow x^2 = 48 \rightarrow x = \sqrt{48}$, or $4\sqrt{3}$. The area of ΔPQT is $1/2 \times 12 \times 4\sqrt{3} = 24\sqrt{3}$ square units. Triangle PSU is congruent, so that's another $24\sqrt{3}$ square units. The length of both TR and UR is $12 - 4\sqrt{3}$ units, so the area of ΔTRU is $1/2 \times (12 - 4\sqrt{3})^2 = 1/2(144 - 96\sqrt{3} + 48) = 1/2(192 - 96\sqrt{3}) = 96 - 48\sqrt{3}$ square units. The area of the shaded area PTU must be $144 - 24\sqrt{3} - 24\sqrt{3} - (96 - 48\sqrt{3}) = 144 - 48\sqrt{3} - 96 + 48\sqrt{3} =$ **48** square units. Note that this is exactly one-third of the square.

Problem 4. The five natural numbers must have a sum of 5 × 6 = 30, and one of the numbers must be 6, the median. The only possible sets of numbers are {2, 6, 6, 8, 8}, {3, 3, 6, 9, 9}, {3, 4, 6, 8, 9}, {3, 5, 6, 7, 9}, {3, 6, 6, 6, 9} and {4, 4, 6, 6, 10}. The sum of all possible largest numbers in the sets is 8 + 9 + 10 = **27**.

Problem 5. In general, the sum of the reciprocals of two numbers is equal to the ratio of the sum to the product. For x and y, we have $1/x + 1/y = y/xy + x/xy = (x + y)/xy$. In a trinomial, $ax^2 + bx + c$, the expression b/a is equal to −1 times of the sum of the roots, and the expression c/a is the product of the roots. Thus, without factoring or using the quadratic formula and without knowing the solutions, we can find the sum of the reciprocals of the roots. It is $(-b/a)/(c/a) = -b/c$, which in this case is −(−3/8) or **3/8**.

Problem 6. The value of P is **1**. Here is the entire grid. The smaller numbers represent one possible order in which the grid could be filled in.

1	4^{1}	3	2^{2}
3	2	4^{5}	1^{4}
4^{6}	1^{7}	2	3^{3}
2	3	1	4



Problem 7. The four corners of the large square can be folded to the center, exactly covering the small square. The ratio of the area of the small square to the area of the large square is **1/2**.

Problem 8. The area of the rectangle with dimensions $(m + 2)$ and $(n - 2)$ is $mn - 2m + 2n - 4$. Since this area is equal to the area of the m-by-n rectangle, we can write the equation $mn = mn - 2m + 2n - 4$. It must be the case that $-2m + 2n - 4 = 0$, which means that $-2m + 2n = 4$, or $-m + n = 2$. Similarly, the area of the rectangle with dimensions $(m - 2)$ and $(n + 10)$ is $mn + 10m - 2n - 20$. This area is also equal to mn, so $10m - 2n - 20 = 0$, which means that $10m - 2n = 20$, or $5m - n = 10$. Adding these two equations, the ns go away, and we have $4m = 12$, which means that $m = 3$. Substituting the 3 for m in the equation $5m - n = 10$, we get $15 - n = 10$, which means that $n = 5$. The area of each rectangle must be 3 × 5 = **15** square units.

Problem 9. If minutes on Ben's clock are the same as minutes on a regular clock, there still would be 12 × 60 = 720 minutes in half a day. These would be divided equally among 10 "hours," so each "Ben-clock hour" would have **72** minutes.

Problem 10. There is only 1 way to build the cube so that all pairs of opposite faces are the same color. (Another way to say this is that no two adjacent faces have the same color.) There are 0 ways to build the cube so that exactly two pairs of opposite faces are the same color. If we have exactly one pair of opposite faces with the same color, those faces could be red, yellow or green. Once the color for the opposites is chosen, there is only 1 way to assemble the other four faces; there would be two adjacent faces of one color and two adjacent faces of the other color. That makes 3 cubes with exactly one pair of opposite faces the same color. Finally, if we insist that no pair of opposite faces has the same color, there are actually 2 ways to build the cube. The cubes would each have three sets of adjacent faces of the same color, but they are mirror images of each other and, therefore, not the same. (We could call one of them "left-handed" and the other one "right-handed.") In all, there are 1 + 3 + 2 = **6** cubes that can be built.

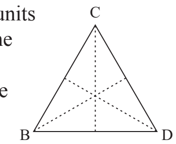
Workout 9

Problem 1. The sum of the integers from 1 to 9 is 45. For the sum of all three sides, the vertex values are used twice because each vertex contributes to two of the sums. The sum S is then (45 + V)/3 = 15 + V/3, where V = the sum of the three vertices. According to the last equation, the sum of the three vertices must be a multiple of 3. Six and 9 have already been used, so the only number that can go be the other vertex is 3, bringing the sum of the three sides to 63. This will be split equally among the three sides of the triangle, so the sum S must be **21**.

Problem 2. The length of segment PQ is the same for all original trapezoid heights, so assume that $h = 4$. The length of segment PQ is also the same regardless of the position of the top line of 6 units with respect to the bottom line of 10 units, so assume that the right endpoints of these segments are directly above one another. The left side of the original trapezoid is then the line $y = x$, with both x and y between 0 and 4 inclusive. The area of the original trapezoid is 8 × 4 = 32. The area of the new trapezoid is 3/4 of this, or 24. The length of segment PQ is $6 + (4 - x) = 10 - x = 10 - y$. The length of the segment halfway between segment PQ and the lower base is then $10 - y/2$. The height of the new trapezoid is y. The area of 24 then equals $(10 - y/2)(y)$. Multiplying this out gives $y^2 - 20y + 48 = 0$. The root between 0 and 4 is $y = 10 - 2\sqrt{13}$.
So PQ = $10 - y =$ $\mathbf{2\sqrt{13}}$ units.

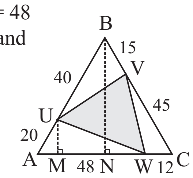
Problem 3. Imagine 20 stones lined up in a row. If we place 2 sticks at random between stones, we will have divided the stones into three groups with a sum of 20. In the end, we have lined up 22 items, 2 of which are sticks. There are "22 choose 2," or 231 ways to do this, but this allows the possibility of zeros in any of the addends. Since this problem requires natural numbers, we should set aside 3 stones, one for each group, and then play the same game with 17 stones and 2 sticks. This is 19 items and we are choosing 2 of them to be sticks. That's "19 choose 2," or $19 \times 18 \div 2 =$ **171** ways.

Problem 4. The volume of a pyramid is one-third the area of the base times the altitude of the pyramid. The area of base BCD is $(1/2) \times 6 \times 3\sqrt{3} = 9\sqrt{3}$ square mm. Since the lateral sides of this pyramid are all congruent, the vertex of the pyramid is directly above the intersection of the medians of base BCD. This point is called the centroid, and it cuts each median in a 1:2 ratio. The centroid is therefore $(1/3) \times 3\sqrt{3} = \sqrt{3}$ units from each side. This is one of the legs of a right triangle whose other leg is the altitude of the pyramid. This right triangle is in the interior of the pyramid, and its hypotenuse is the slant height of one of the lateral sides of the tetrahedron. Since the lateral sides are all isosceles right triangles, this slant height is just $6 \div 2 = 3$ units. Now we can find the altitude x of the pyramid by using the Pythagorean Theorem as follows: $(\sqrt{3})^2 + x^2 = 3^2 \rightarrow 3 + x^2 = 9 \rightarrow x^2 = 6 \rightarrow x = \sqrt{6}$. The volume of tetrahedron ABCD is $(1/3) \times 9\sqrt{3} \times \sqrt{6} = 3\sqrt{18} =$ **$9\sqrt{2}$** cubic millimeters.



Problem 5. A base 2 number with only three digits of 1 must be the sum of three powers of 2. For example, the base 10 number 7 is equal to $4 + 2 + 1$ and is written as 111 in base 2. (Recall that $2^0 = 1$.) Since 2011 is less than $2^{11} = 2048$, we have all powers of 2 from 2^0 to 2^{10} to work with. Any number that is the sum of exactly three of these will have three digits of 1 when it is written in base 2. There are "11 choose 3," or $(11 \times 10 \times 9)/(3 \times 2 \times 1) =$ **165** numbers.

Problem 6. We will start with the area of triangle ABC and subtract the areas of the three nonshaded triangles to get the area of triangle UVW. The height of triangle ABC is $30\sqrt{3}$ cm, and the area of triangle ABC is $(1/2) \times 60 \times 30\sqrt{3} = 900\sqrt{3}$ square cm. If we use AW = 48 as the base of triangle AUW, then the height is $(1/3) \times 30\sqrt{3} = 10\sqrt{3}$. This is because a height dropped from U to segment AC and a height dropped from B to segment AC will form similar triangles with A as a vertex. Therefore, the 20:60 ratio of AU to AB must hold for the two heights. The area of AUW is $(1/2) \times 48 \times 10\sqrt{3} = 240\sqrt{3}$ square cm. If we use BU as the base of BUV, then the height is $(1/4) \times 30\sqrt{3} = (15\sqrt{3})/2$. The area of triangle BUV is $(1/2) \times 40 \times (15\sqrt{3})/2 = 150\sqrt{3}$ square cm. If we use CV as the base of triangle CVW, then the height is $(1/5) \times 30\sqrt{3} = 6\sqrt{3}$. The area of CVW is $(1/2) \times 45 \times 6\sqrt{3} = 135\sqrt{3}$ square cm. Finally, the area of ABC is $900\sqrt{3} - 240\sqrt{3} - 150\sqrt{3} - 135\sqrt{3} = (900 - 240 - 150 - 135)\sqrt{3} =$ **$375\sqrt{3}$** square centimeters.



Problem 7. Let's say that John drove D miles to his uncle's house at an average speed of R miles per hour and it took T hours. This gives us our base equation of D = RT. The two hypothetical situations give us two more equations: D = (R – 10)(T + 2) and D = (R + 20)(T – 2). Expanding these equations, we get D = RT – 10T + 2R – 20 and D = RT + 20T – 2R – 40. If we add these two equations, we get 2D = 2RT + 10T – 60. The 2D on the left side of this equation is equal to the 2RT on the right side (twice our base equation), so it must be the case that 10T = 60, which means T = 6. Plugging this value into each of our earlier equations, we get D = 8R – 80 and D = 4R + 80. Equating these two, we find that 4R = 160, which means that R = 40. So, John actually drove at an average speed of 40 miles per hour for 6 hours, which yields a distance of $40 \times 6 = 240$ miles. To confirm this, we can check that 30 mph for 8 hours is still $30 \times 8 = 240$ miles, and 60 mph for 4 hours is also $60 \times 4 =$ **240** miles.

Problem 8. If we let the two-digit number be digit a followed by digit b, then we have $10a + b = c(a + b)$. We want to find k, such that $10b + a = k(a + b)$. Distributing c in the first equation, we get $10a + b = ca + cb$. If we collect the as on the left and the bs on the right, we get $10a - ca = cb - b$, which can be restated as $a(10 - c) = b(c - 1)$. Solving now for a, we get $a = b(c - 1)/(10 - c)$. Similarly, the original equation with k can be restated as $a = b(10 - k)/(k - 1)$. Since we have two expressions for a, they must be equal to each other, which gives us the equation $b(c - 1)/(10 - c) = b(10 - k)/(k - 1)$. The cross product of this is $b(c - 1)(k - 1) = b(10 - k)(10 - c)$. If we eliminate the bs and expand the remaining expressions, we get $ck - c - k + 1 = 100 - 10c - 10k + ck$. This simplifies to $9k = 99 - 9c$ and then $k =$ **11 – c**, which is the desired expression.

Problem 9. Since 65^2 is the sum of two squares, let's subtract known squares less than or equal to 65^2 and see which one(s) give a difference that is also a perfect square.
$65^2 - 65^2 = 0$; from this we could have the ordered pairs (65, 0), (−65, 0), (0, 65) or (0, −65).
$65^2 - 64^2 = 129$; not a perfect square, but notice that 129 is larger than 11^2. Since subtracting any perfect square less than 11^2 would give a difference between 65^2 and 64^2, we know that the difference won't be a perfect square and thus we don't need to check anything below 11^2.
$65^2 - 63^2 = 256 = 16^2$; from this we have the ordered pairs (63, 16), (63, −16), (−63, 16), (−63, −16), (16, 63), (16, −63), (−16, 63) and (−16, −63).
$65^2 - 62^2 = 381$; not a perfect square
$65^2 - 61^2 = 504$; not a perfect square
$65^2 - 60^2 = 625 = 25^2$; from this we have the ordered pairs (60, 25), (60, −25), (−60, 25), (−60, −25), (25, 60), (25, −60), (−25, 60) and (−25, −60).
$65^2 - 59^2 = 744$; not a perfect square
$65^2 - 58^2 = 861$; not a perfect square
$65^2 - 57^2 = 976$; not a perfect square
$65^2 - 56^2 = 1089 = 33^2$; from this we have the ordered pairs (56, 33), (56, −33), (−56, 33), (−56, −33), (33, 56), (33, −56), (−33, 56) and (−33, −56).
$65^2 - 55^2 = 1200$; not a perfect square
$65^2 - 54^2 = 1309$; not a perfect square
$65^2 - 53^2 = 1416$; not a perfect square
$65^2 - 52^2 = 1521 = 39^2$; from this we have the ordered pairs (52, 39), (52, −39), (−52, 39), (−52, −39), (39, 52), (39, −52), (−39, 52) and (−39, −52).
$65^2 - 51^2 = 1624$; not a perfect square
$65^2 - 50^2 = 1725$; not a perfect square
$65^2 - 49^2 = 1824$; not a perfect square
$65^2 - 48^2 = 1921$; not a perfect square
$65^2 - 47^2 = 2016$; not a perfect square
$65^2 - 46^2 = 2109$; not a perfect square
$65^2 - 45^2 = 2200$; not a perfect square; the square root of 2200 is a little more than 46^2, so we know we have exhausted the possible choices of perfect squares.

That is **36** possible ordered pairs.

Problem 10. Only the units digits of factors contribute to the units digits of products. The same is true for addends and sums. This means we only need to consider the units digit of n and the units digit of 2008 in the expression $n^2 + 2008n$. Clearly, n cannot be odd or have a zero in the units place. If $n = 2$, then $B = 2^2 + 2008 \times 2 = 4 + 4016 = 4020$. If $n = 4$, then $B = 4^2 + 2008 \times 4 = 16 + 8032 = 8048$. If $n = 6$, then $B = 6^2 + 2008 \times 6 = 36 + 12{,}048 = 12{,}084$. There's the 4 we wanted in the units place, and the value of the tens digit is 8. Let's consider $n = 8$ just to be sure. If $n = 8$, then $B = 8^2 + 2008 \times 8 = 64 + 16{,}064 = 16{,}128$, which does not end in a 4. If we were to try $n = 12$, 14, 16 and 18, the units digit results would be the same and only $n = 16$ would give us a number that ends in 4. It also happens that for $n = 6$, 16, 26, 36 and so on, the tens digit is an 8. Thus, the largest possible digit that can be in the tens place of B is **8**.

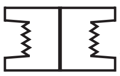
Solids Stretch

Problem 1. When revolved around line segment AB, the right triangle shown will form a **cone**. The side perpendicular to side AB will form the base.

Problem 2. When revolved around line segment AB, the circle will make a full 360-degree revolution forming a **donut** shape (or **torus**).

Problem 3. When revolved around its **diameter**, a **half-circle** or **circle** will form this sphere. Answers may vary.

Problem 4. The line of symmetry in this figure runs through the center of the bases and through the center column (with the zig-zag edges). Thus, the two-dimensional figure that would form the three-dimensional figure shown when revolved about an axis of revolution (the line that runs from the center of one base to the center of the other base) would look something like this:

 or . Answers may vary.

Problems 5a and b. The cylinder revolved about **segment BC** will have the larger volume and surface area since its radius is greater. Both the volume and the surface area involve r^2 and h, so the value of r will have more effect. We prove this to be true when we calculate the ratio of volumes as follows:

$$\frac{V_{BC}}{V_{AB}} = \frac{(\pi \times 12^2) \times 8}{(\pi \times 8^2) \times 12} = \frac{1152\pi}{768\pi} = \frac{3}{2}$$

Using the same thought process as above, we find the ratio of surface areas as follows:

$$\frac{S_{BC}}{S_{AB}} = \frac{2(\pi \times 12^2) + (24\pi)(8)}{2(\pi \times 8^2) + (16\pi)(12)} = \frac{288\pi + 192\pi}{128\pi + 192\pi} = \frac{480\pi}{320\pi} = \frac{3}{2}$$

Problem 6. Using the same logic as in Problem 5a, but using variables instead of numbers, we find the ratio as follows: $\frac{V_b}{V_a} = \frac{\pi \times a^2 \times b}{\pi \times b^2 \times a} = \frac{a}{b}$

Problem 7. Using the same logic as in Problem 5b, but using variables instead of numbers, we find the ratio as follows:

$$\frac{S_b}{S_a} = \frac{2\pi(a) \times b + 2\pi a^2}{2\pi(b) \times a + 2\pi b^2} = \frac{2\pi a(b+a)}{2\pi b(a+b)} = \frac{a}{b}$$

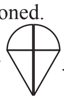
Problem 8. The equation for the volume of a cone is $(1/3)\pi r^2 h$. So when we set up a ratio for the volume of cone A to the volume of cone B, the constants will cancel out, leaving us with $r_A{}^2 h_A / r_B{}^2 h_B$. Thus, the ratio of the volumes is $(c)^2(2c)/(2c)^2(c) = 2c^3/4c^3 =$ **1/2**.

Problem 9a. When the triangle is revolved around side AC, side BC is a radius of the base and AC is the height of the cone. Thus, the volume of the cone is $(1/3)(7^2)\pi(24) =$ **392π** cubic units.

Problem 9b. When the triangle is revolved around segment BC, side AC is a radius of the base and BC is the height of the cone. Thus, the volume of the cone is $(1/3)(24^2)\pi(7) =$ **1344π** cubic units.

Problem 9c. If we look at the area of the triangle in two ways, letting h be the altitude to the hypotenuse, $(1/2)25h = (1/2)7 \times 24 \rightarrow h = 168/25$. The volume of the "double cone" is $(1/3)\pi(168/25)^2 \times 25 =$ **9408π/25** cubic units.

Problem 10a. The line of symmetry in the ice cream cone shown runs from the vertex of the cone through the farthest point on the edge of the dome of ice cream. Thus, the two-dimensional figure shown below would form the ice cream cone when revolved about the line of symmetry mentioned.

 or . Answers may vary.

Problem 10b. Since the cone is packed full of ice cream before the hemisphere is placed on top, we know that the volume of ice cream in the cone is equivalent to the volume of the cone. Since we are looking for the height that would make the volume in the cone equal to the volume outside of the cone, we can set up the following equation: $(1/2)(4/3)\pi 4^3 = (1/3)\pi 4^2 h$. Solving for h, we find that the height of the cone must be **8** cm.

Sum and Product SUPER Stretch

Problem 1. We can factor and solve the equation and then add together the solutions: $(2x - 1)(3x + 4) = 0$, so $x = 1/2$ or $x = -4/3$. Then $1/2 + -4/3 =$ **−5/6**. But there is a useful result from algebra that produces the answer instantly: the Sum and Product Theorem. It states that if $a \neq 0$ and $ax^2 + bx + c = 0$ has the solutions $x = r$ and $x = s$, then $r + s = -b/a$ and $rs = c/a$. From the Sum and Product Theorem, the sum of the solutions of $6x^2 + 5x - 4 = 0$ is $-5/6$.

Problem 2. We could write the factors of the quadratic expression, $(x - (3 + 2\sqrt{2}))(x - (3 - 2\sqrt{2}))$, and multiply to obtain an expression of the form $x^2 + kx + m$, but it is easier to use the Sum and Product Theorem. Adding the solutions $3 + 2\sqrt{2}$ and $3 - 2\sqrt{2}$ gives us 6, so $k = -6$. Multiplying, $(3 + 2\sqrt{2})(3 - 2\sqrt{2}) = 9 - 8 = 1$ and so $m = 1$. Therefore, the value of $k + m = -6 + 1 = \mathbf{-5}$.

Problem 3. Call the solutions of $4x^2 - 13x + 3 = 0$ r and s. Then $r + s = 13/4$ and $rs = 3/4$. We need the value of $1/r + 1/s$, and $1/r + 1/s = (r + s)/(rs)$, so $1/r + 1/s = (13/4)/(3/4) = \mathbf{13/3}$. An alternative solution is also instructive. If we divide each side of the original equation by x^2, we obtain $4 - 13/x + 3/x^2 = 0$, or equivalently, $3(1/x)^2 - 13(1/x) + 4 = 0$. If we let $u = 1/x$, we obtain the equation $3u^2 - 13u + 4 = 0$, and its solutions are the reciprocals of those of $4x^2 - 13x + 3 = 0$. Applying the Sum and Product Theorem to $3u^2 - 13u + 4 = 0$, we find that the sum of the reciprocals of the solutions of the original equation is 13/3.

Problem 4. We know that $r + s = -9/2$ and $rs = 3/2$. So $(r + s)^2 = 81/4$, and thus $r^2 + 2rs + s^2 = 81/4$. Then $r^2 + s^2 = 81/4 - 2rs = 81/4 - 2(3/2) =$ **69/4**.

Problem 5. We know that $r + s = -6$ and $rs = -2$. Then $(r + s)^3 = (-6)^3 = -216$, or equivalently, $r^3 + 3r^2s + 3rs^2 + s^3 = -216$. Therefore, $r^3 + s^3 = -216 - (3r^2s + 3rs^2) = -216 - 3rs(r + s) = -216 - 3(-2)(-6) = \mathbf{-252}$.

Problem 6. Let r and s be the solutions of $x^2 + 7x + 3 = 0$. Then $r + s = -7$ and $rs = 3$. The solutions of $x^2 + bx + c = 0$ are $r + 5$ and $s + 5$, whose sum is $r + s + 10 = -7 + 10 = 3$ and whose product is $(r + 5)(s + 5) = rs + 5(r + s) + 25 = 3 - 35 + 25 = -7$. Therefore, $(b, c) = \mathbf{(-3, -7)}$.

Problem 7. The cubic equation that has solutions $x = 3$, $x = 4$ and $x = 5$ is $(x - 3)(x - 4)(x - 5) = x^3 - 12x^2 + 47x - 60 = 0$. Notice that $3 + 4 + 5 = 12$ and $(3)(4)(5) = 60$. Where does the x coefficient of 47 come from? Notice that $(3)(4) + (3)(5) + (4)(5) = 47$. So the coefficient of x^2 is the opposite of the sum of the solutions, the coefficient of x is the sum of all products of the solutions taken two at a time, and the constant term is the opposite of the product of the solutions. More generally, we have the Sum and Product Theorem for cubics: If $a \neq 0$ and $ax^3 + bx^2 + cx + d = 0$ has solutions r, s and t, then $r + s + t = -b/a$, $rs + rt + st = c/a$ and $rst = -d/a$. By either method, we obtain $(b, c, d) = \mathbf{(-12, 47, -60)}$.

If we'd known the Sum and Product Theorem for cubics before starting the problem, we could have taken a much different approach. From the solutions 3, 4 and 5, we know $d = -(3)(4)(5) = -60$ and $b = -(3 + 4 + 5) = -12$. Substituting these values, as well as $x = 3$ (since 3 is a solution for x), into the original equation, we have $(3)^3 + (-12)(3)^2 + 3c + (-60) = 0 \rightarrow 27 - 108 + 3c - 60 = 0 \rightarrow 3c = 141 \rightarrow c = 47$. Our answer then is $(-12, 47, -60)$

Problem 8. Call the solutions of $x^3 - 3x^2 - 13x + 15 = 0$ r, s and t. Then $r + s + t = 3$, $rs + rt + st = -13$ and $rst = -15$. So the value of $1/r + 1/s + 1/t = st/rst + rt/rst + rs/rst = (st + rt + rs)/rst = -13/-15 = \mathbf{13/15}$.
Or, using the same alternative method as used in Problem 3, dividing each side of the original equation by x^3 yields $1 - 3/x - 13/x^2 + 15/x^3$, or $1 - 3(1/x) - 13(1/x)^2 + 15(1/x)^3$. If we let $u = 1/x$, we obtain the equation $15u^3 - 13u^2 - 3u + 1 = 0$, and its solutions are the reciprocals of the solutions of $x^3 - 3x^2 - 13x + 15 = 0$. Thus, the sum of the solutions of $15u^3 - 13u^2 - 3u + 1 = 0$ is 13/15.

Another method for solving this problem is to see that we must find the three solutions whose product is -15 and whose sum is 3. The factors of 15 are 1, 3, 5 and 15. We need three factors to multiply to 15, and we know one or three of our three factors must be negative. The combination that works is $1, -3, 5$ since $1 - 3 + 5 = 3$. The sum of the solutions' reciprocals is $1/1 - 1/3 + 1/5 = 13/15$.

Problem 9. Call the solutions r, s and t. Then $r + s + t = 15$, $rs + rt + st = 66$ and $rst = 80$. So $(r + s + t)^2 = 225$.
Since $(r + s + t)^2 = r^2 + s^2 + t^2 + 2rs + 2rt + 2st$, we have $r^2 + s^2 + t^2 = 225 - 2(66) = \mathbf{93}$.

Problem 10. Since the solutions of $x^3 - 63x^2 + cx - 1728 = 0$ form a geometric sequence, let's call the solutions m, mr, mr^2. Then $m + mr + mr^2 = 63$ and $(m)(mr)(mr^2) = m^3r^3 = 1728$. Therefore $mr = \sqrt[3]{1728} = 12$. Since c is the sum of all products of pairs of the solutions, $c = (m)(mr) + (m)(mr^2) + (mr)(mr^2) = m^2r + m^2r^2 + m^2r^3 = mr(m + mr + mr^2) = 12(63) = \mathbf{756}$.