Workout 1

Answers

1. 51 (C, F, T)
2. 158,400 (C, F, S, T)
3. 78 (C, F, P, S, T)
4. $3\sqrt{2}, 2\sqrt{6}, 5, \sqrt{27}$ (C, T)
5. 1318 (C, F, P, S, T)
6. B (C, F, P, S, T)
7. 14 (C, F, G, P, T)
8. 26 (C, F, P, T)
9. 3 (C, E, G, P, T)
10. 96 (C, F, P, S, T)

Solution/Representation - Problem #10

We are told that we can make 86,400 periods of 1 second out of the total number of seconds in a day. Notice, we can then make 1 period of 86,400 seconds. Similarly, we are told we can make 43,200 periods of 2 seconds, which means we also can make 2 periods of 43,200 seconds. There will be a symmetry in our listing of possibilities. If we factor 86,400, we see 86,400 = $2^7 \times 3^3 \times 5^2$, which will make it easier to determine which numbers divide into 86,400 evenly and therefore could be a number of periods. Remember, we really only need to find the first half of the options since there is the symmetry identified above. (Extension: We said we only have to test for possible factors until we find the first half of them. Therefore, we only have to test a limited number of integers to see if they are factors of 86,400. How can we determine the largest number we'll have to test? Hint... it's much less than 86,400 ÷ 2 = 43,200.)

Here's a shortcut... The problem didn't ask us to actually list out the possibilities. We just need to know how many possibilities there are. Rather than testing the integers 1 through 86,400 (or the positive integers less than the square root of 86,400) to see if they are factors of 86,400, we can quickly determine the number of factors of 86,400. From above, we have the factorization 86,400 = $2^7 \times 3^3 \times 5^2$. From here we can quickly know the total number of factors. First, take the exponent of each of the prime factors and increase each of them by 1. This gives us 8, 4 and 3. Now find the product of these new numbers: 8 × 4 × 3 = 96. The number 86,400 has **96** factors, and each of these factors could be the number of periods our day could be divided into. *(A more detailed explanation of why this works is included in the solution to Workout 4, Problem #6 in the back of the book.)*

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Workout 2

Answers

1. Lexi; 16 (M, T)
2. 2.8 (C, F)
3. 85.32 (C)
4. 64 (C, M, S)
5. 1990 (C, E, G, T)
6. 25 (C, F, P, T)
7. 8 (C, F, M)
8. 100 (G, P, T)
9. 425 (C, G, P, S, T)
10. 301 (C, E, F, P, T)

Solution/Representation - Problem #8

$	&	»
Ω	©	¢
¤	◊	25

For the addition square shown to the right, we need to determine the value of each of the eight symbols we've placed. According to the guidelines in Problem #8, we know ¤ + ◊ = 25 and » + ¢ = 25. So the sum of the five squares in the last column and last row is (¤ + ◊) + (» + ¢) + 25 = 25 + 25 + 25 = 75. The first row gives us $ + & = » and the second row gives us Ω + © = ¢ . Combining the left sides and the right sides of these two equations we have ($ + &) + (Ω + ©) = » + ¢. Remember, » + ¢ = 25, so ($ + &) + (Ω + ©) = 25. Adding the value of these four squares to the value of the first five we calculated brings our overall sum to 25 + 75 = **100**.

The solution above demonstrates that any solution to the addition square will result in an overall sum of 100. However, if you were unable to think of the steps above, simple trial and error (or Guess & Check, as some might call it) could work, too. Filling in the bottom row and right column is probably the place to start. Two possibilities are shown here. Can you find two more that would work?

6	6	12
6	7	13
12	13	25

5	15	20
5	0	5
10	15	25

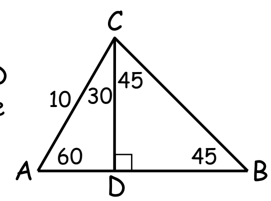
Workout 3

Answers

1. 6 (C, G, P)
2. 128π (C, F, M, T)
3. 959 (C, P, T)
4. 2778 (C, F)
5. 44.7 (C, F, M)
6. 5051 (C, F, M, P, T)
7. 7.4 (C, F)
8. 900 (C, F, G, P)
9. 632,000 (C, F, S)
10. $5+5\sqrt{3}$ (C, F, M)

Solution/Representation - Problem #10

If the measure of angle A is 60° and the measure of angle B is 45°, then we know the measure of angle C is 75° in order to complete the triangle's interior angle sum of 180°. With this information we're a little stuck. However, usually we like angles of 45° and 60° in triangles, especially if they are in 45-45-90 or 30-60-90 triangles. Triangle ABC is not one of these triangles, but notice that dropping an altitude from angle C creates a 90° angle. Triangle ACD is now a 30-60-90 triangle, and triangle BCD is a 45-45-90 triangle. In triangle ACD, AD is the short leg since it is opposite the 30° angle. The short leg in a 30-60-90 is always half the length of its hypotenuse, so AD = 5. The length of the long leg is always the product of $\sqrt{3}$ and the short leg, so CD = $5\sqrt{3}$. Because triangle BCD is 45-45-90, we know CD = BD = $5\sqrt{3}$. Using segment addition, we have AB = AD + DB = **$5+5\sqrt{3}$** units.



Using some trigonometry we can solve this problem a different way. Notice:

sin 75 = sin (30+45) = (sin 30)(cos 45) + (cos 30)(sin 45) = $\frac{1}{2}\times\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}=\frac{\sqrt{2}+\sqrt{6}}{4}$.

Now that we have sin 75, we can solve the equation AC ÷ (sin B) = AB ÷ (sin C) for AB:

$(10)\div\frac{\sqrt{2}}{2}=(AB)\div\frac{\sqrt{2}+\sqrt{6}}{4}$; $\frac{10(2)}{\sqrt{2}}=\frac{4(AB)}{\sqrt{2}+\sqrt{6}}$; $AB=\frac{5(\sqrt{2}+\sqrt{6})}{\sqrt{2}}$; $AB=5+5\sqrt{3}$ units.

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Workout 4

Answers

1.	1.7	(C, F, M)	5.	7.8	(C, F, M)	8.	229	(C)
2.	5	(C, T)	6.	8	(P, T)	9.	14.20	(C, T)
3.	13	(C, P, S, T)	7.	0.5	(C, F, M)	10.	16	(C, G, P)
4.	69	(C, F, M)						

Solution/Representation - Problem #10

We are given that (x)(y) = 3x + 4y and knowing that x and y are both positive integers, we are asked to find the largest possible value of x. Let's get everything on the left side of the equation. We then have xy - 3x - 4y = 0. You may notice that we can add another term to both sides of the equation so that we can factor the left side. See how (x - 4)(y - 3) will get us the first three terms, but will also require a +12 to be included? Adding 12 to both sides we have xy - 3x - 4y + 12 = 12 and so (x - 4)(y - 3) = 12. Remember that we are limited to integer values for x and y, so the values of (x - 4) and (y - 3) will be integers. The factor pairs for 12 are only 1&12, 2&6 and 3&4 -- factor pairs with negative integers will not allow for positive values for x and y. Remember that we want the greatest possible value for x, so we will want to make (x - 4) = 12, 6 or 4 for the three factor pairs. This requires x = 16, 10 or 8. (The corresponding y values would be 4, 5 or 6, respectively.) The greatest possible value for x is **16**.

Let's go back to the equation xy = 3x + 4y. We want the greatest possible integer value of x, so we will solve for x. Getting the terms with an x on the left, we have xy - 3x = 4y. Factoring out the x gives us x(y - 3) = 4y. Now divide both sides by (y - 3) and we have $x = \frac{4y}{y-3}$. Because x must be positive, the smallest possible value for y is 4. This gives us x = 16. Upon investigation of other values of y, we see that as y increases, the value of x decreases, so x = **16** is the greatest possible value of x.

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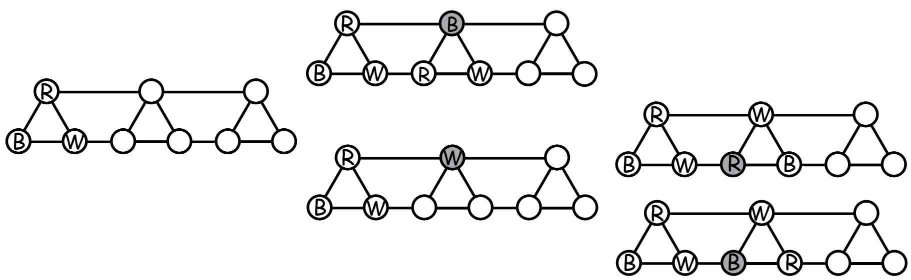
Workout 5

Answers

1. 15,625 (C, G, P, T)
2. 4.272 (C, F, M)
3. 65.9 (C, F)
4. 1,090,000 (C, F, M)
5. 3.1 (C, F, M)
6. $\frac{1}{4}$ (C, F, M, S)
7. 54 (M, P, S, T)
8. 45 (E, G, T)
9. 5 (C, E, P, T)
10. 356,400 (C, F, M)

Solution/Representation - Problem #7

Let's start with the triangle farthest to the left. We'll label its three vertices red, white and blue. We know each of its vertices must be a different color since they are directly connected by segments. In the example below, we went with red on top, blue on the bottom left and white on the bottom right. There are actually six different ways we could have colored the first triangle. With one triangle colored, we are more limited in the number of ways the vertices of the second triangle can be colored. In our example, the top vertex can't be red, but it can be blue or white. Notice that if the top vertex is blue, the bottom two vertices can be colored in only one way. However, if we make the top vertex white, there are two options for the bottom vertices. From this we see that when the vertices of a triangle are colored in, the next triangle's vertices will be limited to three options. So, the first triangle has six ways in which its vertices can be colored. Each of those ways can be matched with three options for the second triangle; and then each of those can be matched with three options for the third triangle. This result is $6 \times 3 \times 3 =$ **54** ways to color the vertices in this figure according to the constraints.



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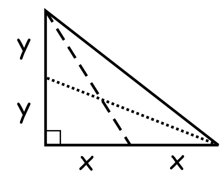
Workout 6

Answers

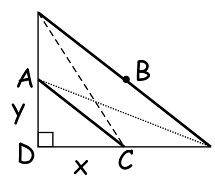
1. 170 (C, F, M, P)
2. 32 (C)
3. 3.6 (C, F, G, M)
4. 300 (C, P, S, T)
5. 96 (F, P, T)
6. 2120 (C, F, M)
7. 9 (G, P)
8. $\frac{4}{9}$ (C, P, T)
9. 280,000,000 (C, F)
10. 17 (C, F, G)

Solution/Representation - Problem #3

A figure may help to visualize the situation. We know a median starts at the vertex of a triangle and intersects the opposite side at its midpoint. Let our right triangle have legs of length 2x and 2y units. The dotted line is the median with length $\sqrt{40}$ units, and the dashed line is the median with length 5 units. Notice each of these medians serves as the hypotenuse of a right triangle, and with the Pythagorean Theorem we can write the equations shown beneath the figure. If we simplify these equations and then add the left sides together and the right sides together, we have $65 = 5x^2 + 5y^2$ or $13 = x^2 + y^2$. We still don't have the median from the right angle that we are looking for. However, notice that this equation $13 = x^2 + y^2$ or $(\sqrt{13})^2 = x^2 + y^2$ would be the equation we would write for right triangle ADC in the figure below, and the length of hypotenuse AC is $\sqrt{13}$. This segment AC is called a midsegment because it joins the midpoints of two sides of a triangle. The midsegment is parallel to the third side, and its length is half the length of the third side. If B is the midpoint of the original hypotenuse, then AB and BC are midsegments, and we can see BC = y units and AB = x units. Quadrilateral ABCD is now a rectangle with diagonals AC and BD. The diagonals of a rectangle are congruent, so median BD = AC = $\sqrt{13}$ ≈ **3.6** units, to the nearest tenth.



$$(\sqrt{40})^2 = y^2 + (2x)^2$$
$$(5)^2 = x^2 + (2y)^2$$



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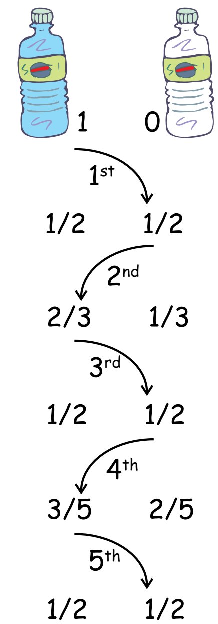
Workout 7

Answers

1. 98 (*C*, E, P, T)
2. 945 (*C*, F, T)
3. 22 (M, P, T)
4. 7 (*C*, P, S, T)
5. 715 (*C*, E, *G*, P, T)
6. 5 (E, *G*, P, T)
7. 0 (E, *G*, P, T)
8. $\frac{1}{2}$ (P, T)
9. 4.88 (*C*, F)
10. 777 (*C*, E, *G*, P, T)

Solution/Representation - Problem #8

The solution to this problem could get very long! Let's see what happens during the first few steps of the pouring process. To the right we can see the progression of the 1^{st} through 5^{th} pours. Notice that for the 2^{nd} pour, we are pouring 1/3 of second bottle's water into the first bottle. Rather than trying to calculate the amount of water that will be in the first bottle after this pour, it is easier to consider that we will be *keeping* 2/3 of the water in the second bottle, or (2/3)(1/2) = 1/3. That means the first bottle will be 1 – 1/3 = 2/3 full. Again, when we go to pour 1/4 of this amount into the second bottle for the third pour, first determine how much will *stay* in the first bottle. We will keep 3/4 of the 2/3, which is (3/4)(2/3) = 1/2. So the second bottle will contain 1 – 1/2 = 1/2 after the pour, too. As this progresses we see that after an "odd" pour our bottles are back to each being half full. We are asked for the amount of water in the first bottle after the 2007^{th} pour, which is an odd pour. So each bottle will be half full, and the first bottle will have **1/2** quarts.



Using the pattern we have started to identify, can we determine a formula for the amount of water in the first bottle after *n* pours if *n* is even? How much water would be in the first bottle after the 2008^{th} pour?

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Workout 8

Answers

1. 0.68 (F, M, S)	5. 30 (E, G, T)	8. 318 (C, P)
2. 3.24 (C, F, S, T)	6. 1764 (F, M)	9. 3 (E, G, T)
3. 30 (E, G, M, P, T)	7. $\frac{3}{14}$ (C, F)	10. 5 (P, S)
4. 516 (C, P, S, T)		

Solution/Representation - Problem #2

We were told the rate at which pairs of hoses could fill the pool: A & B take 4 hrs. for one pool; A & C take 5 hrs. for one pool; B & C take 6 hrs. for one pool. The least common multiple of 4, 5 and 6 is 60, so in 60 hours, A & B can fill 15 pools, A & C can fill 12 pools, and B & C can fill 10 pools. Obviously hose A can't be working on two different pools at the same time, but let's assume we have two of each kind of hose. So in 60 hours (A + B) + (A + C) + (B + C) can fill 37 pools. Rewriting this as 2A + 2B + 2C, we can understand that A + B + C can fill the same 37 pools in 120 hours. If we want to know how long it will take them to fill just one pool altogether we can calculate 120 ÷ 37 ≈ **3.24** hours.

Here's another way to represent this problem. We know that in one hour A & B can fill 1/4 = 30/120 of a pool; in one hour A & C can fill 1/5 = 24/120 of a pool; and in one hour B & C can fill 1/6 = 20/120 of a pool. Notice A & B fill 30/120 - 24/120 = 6/120 more of a pool than A & C and this must be because B does 6/120 more in an hour than C does. So if B & C fill 20/120 of a pool in an hour, and B fills 6/120 more than C, we know C fills 7/120 of a pool in an hour and B fills 13/120 of a pool in an hour. And if A & B fill 30/120 of a pool in an hour, A fills 17/120 in an hour. Together the three hoses would fill 37/120 of a pool in an hour, so they would need 120/37 of an hour to complete the job. This is 120 ÷ 37 ≈ **3.24** hours.

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Workout 9

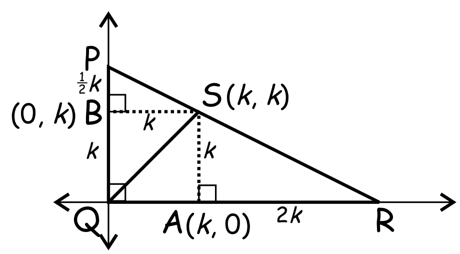
Answers

1. 496 (C, G, M, P, S, T)
2. 37.5 (C, E, G, M, P)
3. 1458 (C, G, P, S)
4. $\frac{10\sqrt{2}}{3}$ (C, F, M)
5. $\frac{35}{72}$ (C, F, M, P, T)
6. 17 (C, E, F, G, M, T)
7. 11 (C, F, G, P)
8. $\frac{3}{10}$ (C, F, M)
9. 236 (C, F, G, M, P, T)
10. LQSX (C, E, G, P)

Solution/Representation - Problem #4

Angle PQR is a right angle, so if segment QS bisects angle PQR, it creates two 45-degree angles. This also means segment QS is on the line $y = x$ and point S is at (k, k) for some value k. The line connecting points P and R has a slope of $-1/2$, a y-intercept of 5 and the equation $y = (-1/2)x + 5$. This line intersects the line $y = x$ at point S. Solving the system of equations we have $x = (-1/2)x + 5$, which leads to $x = 10/3$. So point S is at (10/3, 10/3). The segment QS is the hypotenuse of the 45-45-90 triangle with legs each measuring 10/3 units, and therefore, its length is $(10/3)(\sqrt{2})$ or $(10\sqrt{2})/3$ units.

From the coordinates of P, Q and R, we know PQ = 5 units and QR = 10 units. We also can see that triangle PQR is similar to triangles SAR and PBS, as shown here. This means the ratio of short leg to long leg in triangle PQR, which is 5:10 = 1:2, holds for triangles SAR and PBS. Notice in this figure, we see that PB + BQ = $(1/2)k + k = 5$. Solving for k we get $k = 10/3$. The coordinates of point S are then (10/3, 10/3). QS is the hypotenuse to 45-45-90 triangle SBQ, so its measure is $(10/3)(\sqrt{2})$ or $(10\sqrt{2})/3$ units.



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